

ON THE GEOMETRY OF A THREE-DIMENSIONAL RIEMANNIAN MANIFOLD WITH A SKEW-CIRCULANT STRUCTURE AND AN ASSOCIATED METRIC

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ABSTRACT. A three-dimensional Riemannian manifold M with a metric g and an additional tensor structure ϕ of type $(1, 1)$ is considered. The local components of these two tensors form skew-circulant matrices. The third power of ϕ is the identity, and ϕ is compatible with g .

An associated indefinite metric $\tilde{g}$ on (M, g, ϕ) , defined by g and ϕ , is introduced. It is shown that if the Levi-Civita connection of $\tilde{g}$ is locally flat, then (M, g, ϕ) is an almost-Einstein manifold. For such a manifold, the Ricci curvatures in the direction of the basis vectors of an arbitrary ϕ -basis of $T_p M$ are calculated. Expressions for the curvature tensors R and $\tilde{R}$ are obtained in terms of the metrics g and $\tilde{g}$ and the scalar curvatures. Equalities between the sectional curvatures of special 2-planes determined by the basis vectors of an arbitrary ϕ -basis are established.

The equation of a sphere with respect to the associated metric $\tilde{g}$ is considered. Since $\tilde{g}$ is an indefinite metric, the equation of the sphere transforms into the equations of three types of quadric surfaces embedded in Euclidean space: a cone, a hyperboloid of one sheet, and a hyperboloid of two sheets.

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1. INTRODUCTION

Differentiable manifolds with additional tensor structures have been the subject of intensive and in-depth research over the last few decades. Two main types of additional tensor structures (of the polynomial tensor structure type) are well known: the first type is an *almost product structure*, and the second type is an *almost complex structure*.

Depending on the type of metric with which differentiable manifolds are equipped—Riemannian (positive definite) or pseudo-Riemannian (indefinite)—they are classified as Riemannian or pseudo-Riemannian manifolds, respectively. Depending on how the additional structure is compatible with the metric, there are two types of manifolds with an almost product structure (Riemannian manifolds with an almost product structure and almost para-Hermitian manifolds) and, correspondingly, two types of manifolds with an almost complex structure (almost complex manifolds with a Norden metric and almost Hermitian manifolds).

Riemannian manifolds with an almost product structure have been actively studied in recent years by foreign and Bulgarian geometers, as seen, for example, in the papers, [11, 12, 13, 16, 17, 19, 20], and [21].

Almost complex manifolds with a Norden metric and almost Hermitian manifolds are of even greater interest to modern differential geometry. We list here only a few studies that examine the curvature properties of such manifolds: [1, 5, 7, 8, 9, 10, 18], and [22].

We also mention several studies on three-dimensional Riemannian manifolds in which the third power of the additional structure satisfies other polynomial equalities, such as [23, 24]. In [3], the structure Q is circulant (meaning the matrix of its components is circulant) and satisfies $Q^3 = \text{id}$. In [14], the structure S is skew-circulant (the matrix of its components is skew-circulant) and satisfies $S^3 = -\text{id}$.

In this study, we consider a three-dimensional Riemannian manifold M with a metric g and an additional tensor structure ϕ of type $(1, 1)$. The local components of these two tensors form skew-circulant matrices. The third power of ϕ is equal to the identity, and ϕ is compatible with g such that an isometry is induced on the tangent space $T_p M$ at an arbitrary point p of M .

Our aim is to investigate the geometric properties of such a manifold (M, g, ϕ) . Furthermore, by examining it, we compare its similarities and differences with Riemannian manifolds endowed with an additional circulant structure, which were previously studied by I. Dokuzova, G. Dzhelepov, and D. Razpopov in [2, 3], and [4].

The paper is organized as follows. Section 2 presents basic information, regarding a 3-dimensional Riemannian manifold (M, g, ϕ) , the metric, the Levi-Civita connection, the Riemannian curvature tensor, the Ricci tensor, sectional curvatures, and the Ricci curvature.

In Section 3, the Christoffel symbols of the Levi-Civita connection ∇ induced by the metric g on (M, g, ϕ) are calculated. Differential equations are obtained for the components of the metric such that the structure ϕ is parallel with respect to ∇ . This means that ϕ remains invariant under parallel transport along curves on the manifold. An example of a manifold (M, g, ϕ) with a parallel structure ϕ is constructed.

Section 4 examines an associated indefinite metric $\tilde{g}$ on (M, g, ϕ) , defined by g and the additional structure ϕ . A fundamental tensor F of type $(0, 3)$ is introduced via g and ϕ , as well as the two Lee 1-forms α and α^* associated with F . The components of F are calculated, and a representation of F in terms of the metrics and both 1-forms is obtained. This representation of F allows one to derive a relationship between the Ricci tensors associated with g and $\tilde{g}$. A definition of almost Einstein manifolds is given, which generalizes the definition of Einstein manifolds. If $\tilde{\nabla}$ is a locally flat connection, then (M, g, ϕ) is an almost Einstein manifold; similarly, if ∇ is a locally flat connection, then $(M, \tilde{g}, \phi)$ is also an almost Einstein manifold.

In Section 5, the existence of an orthogonal ϕ -basis in each tangent space $T_p M$ of (M, g, ϕ) is established. For an almost Einstein manifold (M, g, ϕ) , the Ricci curvatures in the direction of the basis vectors of an arbitrary ϕ -basis of $T_p M$ are calculated. Expressions for the curvature tensors R and $\tilde{R}$ are obtained in terms of the metrics g and $\tilde{g}$, and the scalar curvatures. Equalities are found between the sectional curvatures of 2-planes determined by the basis vectors of an arbitrary ϕ -basis.

In Section 6, the equation of a sphere with respect to the associated metric $\tilde{g}$ is considered. This equation is transformed in terms of the Riemannian metric g . Since $\tilde{g}$ is an indefinite metric, it defines spacelike, timelike, and isotropic vectors. Furthermore, the equation of the sphere with respect to $\tilde{g}$ is transformed into the equations of three types of quadric surfaces embedded in Euclidean space. These are a cone, a hyperboloid of one sheet, and a hyperboloid of two sheets.

2. PRELIMINARIES

We present well-known facts and formulas for a Riemannian manifold (introduced, for example, in [15] and [25]). Furthermore, we consider the definition of

a three-dimensional Riemannian manifold (M, g, ϕ) , the geometric properties of which are the subject of the present work.

2.1. Necessary formulas for a Riemannian manifold. On a Riemannian manifold M , the metric g induces a linear connection ∇ – the Levi-Civita connection. If Γ_{ij}^k are the Christoffel symbols of ∇ , then

$$2\Gamma_{ij}^k = g^{ks}(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}). \quad (2.1)$$

In (2.1), the matrix (g^{ks}) is the inverse of $g = (g_{sj})$.

The covariant derivative of a tensor ϕ of type $(1, 1)$ is given by

$$\nabla_k \phi_i^s = \partial_k \phi_i^s + \Gamma_{ka}^s \phi_i^a - \Gamma_{ki}^a \phi_a^s. \quad (2.2)$$

If $\nabla \phi = 0$, then ϕ is a parallel structure with respect to ∇ .

The Riemann curvature tensor R is defined by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z,$$

where x, y and z are smooth vector fields on M . The associated tensor with R of type $(0, 4)$ is given by

$$R(x, y, z, u) = g(R(x, y)z, u).$$

Its local expression is $R_{tijk} = g_{st} R_{ijk}^t$, where the components of R are calculated by

$$R_{ijk}^t = \partial_j \Gamma_{ik}^t + \Gamma_{ik}^\alpha \Gamma_{\alpha j}^t - \partial_i \Gamma_{jk}^t - \Gamma_{jk}^\alpha \Gamma_{\alpha i}^t. \quad (2.3)$$

The Ricci tensor ρ and the scalar curvature τ of M are given by:

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j). \quad (2.4)$$

In this paper, all indices i, j, k, s vary in the set $\{1, 2, 3\}$.

Let p be a point on M , let x and y be linearly independent vectors on the tangent space $T_p M$. The value

$$k(x, y) = \frac{R(x, y, x, y)}{g^2(x, y) - g(x, x)g(y, y)} \quad (2.5)$$

is the sectional curvature of the 2-plane $\{x, y\}$.

2.2. Basic notations for a manifold (M, g, ϕ) . The matrices used to represent the tensor structures are skew-circulant. Notably, these are a subclass of Toeplitz matrices, which have been extensively studied in [6].

In this paper, we focus on the following skew-circulant matrices:

$$\phi = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \phi^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (2.6)$$

They both together with the identity matrix form a basis of the real vector space of the 3-dimensional skew-circulant matrices.

We consider a 3-dimensional Riemannian manifold M with a metric g and an additional tensor structure ϕ , whose components are expressed by the first matrix in (2.6). Then, we obtain

$$\phi^3 = \text{id}, \quad \phi \neq \text{id}. \quad (2.7)$$

Let the structure ϕ and the metric g are compatible, as follows:

$$g(\phi x, \phi y) = g(x, y). \quad (2.8)$$

Consequently, the component matrix of g is also skew-circulant and takes the form

$$(g_{ij}) = \begin{pmatrix} A & B & -B \\ B & A & B \\ -B & B & A \end{pmatrix}. \quad (2.9)$$

Here, $A = A(p)$ and $B = B(p)$ are smooth functions of an arbitrary point $p(x^1, x^2, x^3) \in M$. Furthermore, we assume that conditions

$$A > 2B > 0 \quad (2.10)$$

hold. They are necessary to ensure that the metric g is positive definite.

In the following sections, we investigate the geometric properties of the manifold (M, g, ϕ) . Specifically, in Section 4, we equip this manifold with an associated indefinite metric $\tilde{g}$.

3. MANIFOLDS WITH A PARALLEL STRUCTURE

In this section, we compute the Christoffel symbols for the Levi-Civita connection ∇ induced by the metric g on (M, g, ϕ) . We derive partial differential equations governing the components of g under which the structure ϕ is parallel with respect to ∇ . Finally, we construct a non-trivial example of such a manifold.

Assume that the tensor structure ϕ on (M, g, ϕ) is parallel, meaning that it satisfies the condition

$$\nabla \phi = 0. \quad (3.1)$$

In local coordinates, (3.1) is written as $\nabla_i \phi_j^s = 0$. From (2.2), we get

$$\Gamma_{ia}^s \phi_j^a = \Gamma_{ij}^s \phi_a^s.$$

Then, from (2.6), we derive the next statement:

Lemma 3.1. *Let (M, g, ϕ) have a parallel structure ϕ with respect to the Levi-Civita connection ∇ of g . Then the Christoffel symbols for ∇ satisfy equalities:*

$$\begin{aligned}\Gamma_{13}^1 &= -\Gamma_{11}^2 = -\Gamma_{12}^3 = \Gamma_{23}^2 = \Gamma_{22}^1 = \Gamma_{33}^3, \\ \Gamma_{13}^2 &= -\Gamma_{11}^3 = \Gamma_{12}^1 = \Gamma_{23}^3 = \Gamma_{22}^2 = -\Gamma_{33}^1, \\ \Gamma_{13}^3 &= \Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{23}^1 = \Gamma_{22}^3 = -\Gamma_{33}^2.\end{aligned}\tag{3.2}$$

We obtain a necessary and sufficient condition for the structure ϕ to be parallel in the following theorem:

Theorem 3.2. *The structure ϕ on (M, g, ϕ) satisfies (3.1) if and only if the functions A and B fulfill the system of partial differential equations:*

$$A_1 = B_1 + B_2 - B_3, \quad A_2 = B_1 + B_2 + B_3, \quad A_3 = -B_1 + B_2 + B_3, \tag{3.3}$$

where $A_i = \frac{\partial A}{\partial x^i}$ and $B_i = \frac{\partial B}{\partial x^i}$.

Proof. The inverse matrix of g is given by

$$(g^{ij}) = \frac{1}{D} \begin{pmatrix} A-B & -B & B \\ -B & A-B & -B \\ B & -B & A-B \end{pmatrix}, \tag{3.4}$$

where $D = (A+B)(A-2B)$.

Next, we compute the Christoffel symbols using (2.1), (2.9), and (3.4), yielding the following expressions:

$$\begin{aligned}\Gamma_{ii}^i &= \frac{1}{2D} ((A-B)A_i - B(4B_i + (-1)^j A_k + (-1)^k A_j)), \\ \Gamma_{ii}^k &= -\frac{(-1)^j}{2D} ((A-B)(2B_i + (-1)^j A_k) + B(2B_i + (-1)^k A_j - A_i)), \\ \Gamma_{ik}^k &= \frac{1}{2D} ((A-B)A_i + (-1)^i B((-1)^j B_j - (-1)^i B_i - (-1)^k B_k \\ &\quad + (-1)^j A_k)), \\ \Gamma_{ij}^k &= \frac{1}{2D} ((A-B)((-1)^k B_k - (-1)^i B_i - (-1)^j B_j) \\ &\quad + B((-1)^i A_i + (-1)^j A_j)),\end{aligned}\tag{3.5}$$

where $i \neq j \neq k$.

Assuming that Equality (3.1) holds, substituting (3.5) into (3.2), we obtain the system of differential equations:

$$\begin{aligned} 2A_2 + A_3 &= 3B_1 - B_2 + B_3, & 2A_1 + A_3 &= -B_1 + 3B_2 + B_3, \\ A_1 + 2A_3 &= B_1 + 3B_2 - B_3, & A_1 + 2A_2 &= B_1 - B_2 + 3B_3, \\ 2A_1 + A_2 &= -B_1 + B_2 + 3B_3, & A_2 + 2A_3 &= 3B_1 + B_2 - B_3, \\ A_1 - A_3 &= -2B_1 + 2B_3, & A_2 - A_3 &= -2B_2 + 2B_3, \\ A_1 - A_2 &= -2B_1 + 2B_2. \end{aligned}$$

This system is equivalent to (3.3).

Conversely, substituting (3.3) into (3.5) yields (3.2), where

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2D}((A - B)A_1 - 2BB_1), \\ \Gamma_{11}^2 &= \frac{1}{2D}((A - B)(-A_3) + 2BB_3), \\ \Gamma_{11}^3 &= \frac{1}{2D}((A - B)A_2 - 2BB_2). \end{aligned} \tag{3.6}$$

Taking into account (2.2), a direct calculation shows that $\nabla_i \phi_j^s = 0$; that is, condition (3.1) holds true. $\square$

3.1. Example. We construct an example of the manifold (M, g, ϕ) with a parallel structure ϕ by finding an explicit solution to the system (3.3).

Let g be determined by (2.9) with the functions

$$A = 2((x^1)^2 + (x^2)^2 + (x^3)^2), \quad B = 2x^1x^2 + 2x^2x^3 - 2x^1x^3, \tag{3.7}$$

where $B > 0$. Consequently, inequalities in (2.10) are satisfied. A direct calculation verifies that the functions (3.7) and their derivatives fulfill the system (3.3). Thus, we establish the following proposition:

Proposition 3.3. *The manifold (M, g, ϕ) with the metric g determined by (3.7) admits a parallel tensor structure ϕ .*

Proposition 3.4. *The manifold (M, g, ϕ) with the metric g determined by (3.7) is not flat.*

Proof. Using (2.3), we find a non-zero component of R , for instance R_{212}^1 :

$$R_{212}^1 = \partial_1 \Gamma_{22}^1 + \Gamma_{22}^\alpha \Gamma_{\alpha 1}^1 - \partial_2 \Gamma_{12}^1 - \Gamma_{12}^\alpha \Gamma_{\alpha 2}^1,$$

and using (3.6), we calculate

$$\begin{aligned}
 R_{212}^1 &= -\partial_1 \Gamma_{11}^2 + \partial_2 \Gamma_{11}^3 = -\frac{1}{2D^2} ((-A_3(A_1 - B_1) + 2B_1B_3 - 4B)D \\
 &\quad + (2AA_1 - A_1B - AB_1 - 4BB_1)(-A_3(A - B) + 2BB_3)) \\
 &\quad - \frac{1}{2D^2} ((A_2(A_2 - B_2) + 4(A - B) - B_2^2)D \\
 &\quad + (2AA_2 - A_2B - AB_2 - 4BB_2)(A_2(A - B) - 2BB_2)).
 \end{aligned}$$

A direct calculation shows that at a specific point—for instance, $p(1, 1, 1)$ —the curvature component $R_{212}^1 = \frac{1}{8}$ is non-zero. $\square$

4. AN ASSOCIATED METRIC ON (M, g, ϕ)

In the first part of this section, we introduce an associated metric $\tilde{g}$ on (M, g, ϕ) determined by the structures g and ϕ . Using $\tilde{g}$, we define a fundamental tensor F , which in turn induces two Lee 1-forms. In the second part, we determine the conditions under which the manifold (M, g, ϕ) is almost Einstein or, in particular, Einstein.

We define $\tilde{g}$ via g and ϕ as follows:

$$\tilde{g}(x, y) = g(\phi x, y) + g(x, \phi y). \quad (4.1)$$

By virtue of (2.8) and (4.1), we obtain

$$\tilde{g}(\phi x, \phi y) = \tilde{g}(x, y). \quad (4.2)$$

With the help of (2.6) and (2.9), we find the component matrix of $\tilde{g}$:

$$(\tilde{g}_{ij}) = \begin{pmatrix} -2B & B - A & A - B \\ B - A & -2B & B - A \\ A - B & B - A & -2B \end{pmatrix}. \quad (4.3)$$

This metric is indefinite because, due to (2.10), the minors of the determinant of $\tilde{g}$ have different signs; for example:

$$\tilde{g}_{11} = -2B < 0, \quad \det \tilde{g} = (A + B)^2(A - 2B) > 0.$$

The inverse matrix of $\tilde{g}$ takes the form

$$(\tilde{g}^{ij}) = \frac{1}{2D} \begin{pmatrix} 3B - A & B - A & A - B \\ B - A & 3B - A & B - A \\ A - B & B - A & 3B - A \end{pmatrix}, \quad (4.4)$$

where $D = (A + B)(A - 2B)$.

4.1. Fundamental tensor. The tensor F of type $(0, 3)$ and both Lee 1-forms α and α^* are defined by:

$$\begin{aligned} F(x, y, z) &= g((\nabla_x \phi)y, z), \\ \alpha(k) &= g^{ij}F(e_i, e_j, x), \quad \alpha^*(k) = g^{ij}F(e_i, \phi e_j, x). \end{aligned} \quad (4.5)$$

In local coordinates, the tensor F is given by $F_{kij} = \nabla_k \tilde{g}_{ij}$. By virtue of (4.1), the following symmetry property holds:

$$\nabla_k \tilde{g}_{ij} = \nabla_k \tilde{g}_{ji}. \quad (4.6)$$

Obviously F has the property $F(x, y, z) = F(x, z, y)$.

In the next theorem, we find an explicit expression for F in terms of the metrics and the Lee 1-forms. This representation subsequently enables us to express the Ricci tensors associated by both metrics.

Theorem 4.1. *Let (M, g, ϕ) be a Riemannian manifold with the fundamental tensor F and the Lee 1-forms α and α^* defined by (4.5). Then, the tensor F satisfies*

$$\begin{aligned} F(x, y, z) &= \frac{1}{3} \{ g(x, y)\alpha(z) + g(x, z)\alpha(y) \\ &\quad + \tilde{g}(x, y)\alpha^*(z) + \tilde{g}(x, z)\alpha^*(y) \}. \end{aligned} \quad (4.7)$$

Proof. We apply (2.2) to $\tilde{g}$:

$$\nabla_k \tilde{g}_{is} = \partial_k \tilde{g}_{is} - \Gamma_{ks}^a \tilde{g}_{ia} - \Gamma_{ki}^a \tilde{g}_{as}. \quad (4.8)$$

Then, using (3.5), (4.3), (4.4), (4.5), and (4.6), we compute the components of F :

$$\begin{aligned} F_{111} &= 2F_{123} = 2B_1 - A_2 + A_3, \\ F_{222} &= -2F_{213} = 2B_2 - A_1 - A_3, \\ F_{333} &= 2F_{312} = 2B_3 + A_1 - A_2, \\ F_{122} &= F_{322} = A_2 - B_1 - B_2 - B_3, \\ F_{311} &= -F_{211} = -A_1 + B_1 + B_2 - B_3, \\ F_{233} &= -F_{133} = A_3 + B_1 - B_2 - B_3, \\ F_{212} &= -F_{112} = \frac{1}{2}(B_1 - B_2 - B_3 + A_3), \\ F_{113} &= F_{313} = \frac{1}{2}(B_1 + B_2 + B_3 - A_2), \\ F_{323} &= -F_{223} = \frac{1}{2}(B_1 + B_2 - B_3 - A_1). \end{aligned} \quad (4.9)$$

Next, from (3.4), (4.5), and (4.9), for the components of α we get:

$$\begin{aligned}\alpha_1 &= \frac{3}{2D}((A-B)(A_3-A_2) + 2B(B_2-B_3-A_1) + 2AB_1), \\ \alpha_2 &= \frac{3}{2D}((A-B)(-A_1-A_3) + 2B(B_1+B_3-A_1) + 2AB_2), \\ \alpha_3 &= \frac{3}{2D}((A-B)(A_1-A_2) + 2B(B_2-B_1-A_3) + 2AB_3).\end{aligned}\quad (4.10)$$

For the components of α^* , with the help of (4.4), (4.5), and (4.9), we find:

$$\begin{aligned}\alpha_1^* &= \frac{3}{2D}((A+2B)B_1 + A(B_2-B_3-A_1) + B(A_3-A_2)), \\ \alpha_2^* &= \frac{3}{2D}((A+2B)B_2 + A(B_1+B_3-A_2) + B(-A_1-A_3)), \\ \alpha_3^* &= \frac{3}{2D}((A+2B)B_3 + A(B_2-B_1+A_3) + B(A_1-A_2)).\end{aligned}\quad (4.11)$$

The Equalities (2.9), (4.3), (4.9), (4.10), and (4.11) imply

$$F_{ijk} = \frac{1}{3} (g_{ij}\alpha_k + g_{ik}\alpha_j + \tilde{g}_{ij}\alpha_k^* + \tilde{g}_{ik}\alpha_j^*), \quad (4.12)$$

which is a local form of (4.7). $\square$

4.2. Almost Einstein manifolds. Let $\tilde{\Gamma}$ and $\tilde{\nabla}$ denote the Christoffel symbols and the Levi-Civita connection of the associated metric $\tilde{g}$, respectively. Furthermore, let $\tilde{R}$ be the curvature tensor associated with $\tilde{\nabla}$. The Ricci tensor $\tilde{\rho}$ and the scalar curvature $\tilde{\tau}$ of $\tilde{g}$ are given by:

$$\tilde{\rho}(y, z) = \tilde{g}^{ij} \tilde{R}(e_i, y, z, e_j), \quad \tilde{\tau} = \tilde{g}^{ij} \tilde{\rho}(e_i, e_j). \quad (4.13)$$

By applying (2.1) to both Γ and $\tilde{\Gamma}$, we deduce from (4.8) the following relation:

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2} \tilde{g}^{ks} (\nabla_i \tilde{g}_{js} + \nabla_j \tilde{g}_{is} - \nabla_s \tilde{g}_{ij}).$$

We substitute (4.12) into the above equality and obtain

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{3} \tilde{g}^{ks} (g_{ij}\alpha_s + \tilde{g}_{ij}\alpha_s^*). \quad (4.14)$$

Next, we calculate

$$\tilde{g}^{sk} g_{sj} = L_j^k, \quad g^{sk} \tilde{g}_{sj} = \frac{1}{2} S_j^k,$$

where

$$(L_j^k) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad (S_j^k) = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}.$$

Then, from (4.5), (4.10), and (4.11), we get

$$\begin{aligned}\tilde{g}^{sk}\alpha_s &= -\alpha^{*k}, & \tilde{g}_{sk}\alpha^s &= \alpha_k - 2\alpha_k^*, \\ \tilde{g}_{sk}\alpha^{*s} &= -\alpha_k, & \tilde{g}^{sk}\alpha_s^* &= -\frac{1}{2}(\alpha^k + \alpha^{*k}).\end{aligned}\quad (4.15)$$

In view of (4.14) and (4.15), the components of the affine deformation tensor $T = \tilde{\Gamma} - \Gamma$ are given by:

$$T_{ij}^k = -\frac{1}{6}(2g_{ij}\alpha^{*k} + \tilde{g}_{ij}(\alpha^k + \alpha^{*k})).$$

We substitute this relation into the well-known formula linking R and $\tilde{R}$, which reads:

$$\tilde{R}_{ijs}^k = R_{ijs}^k + \nabla_j T_{is}^k - \nabla_s T_{ij}^k + T_{is}^a T_{aj}^k - T_{ij}^a T_{as}^k.$$

Thus, Equality (4.15) yields

$$\begin{aligned}\tilde{R}_{ijs}^k &= R_{ijs}^k - \frac{1}{3}g_{is}(\nabla_j \alpha^{*k} - \frac{1}{3}\alpha_j^* \alpha^{*k}) + \frac{1}{3}g_{ij}(\nabla_s \alpha^{*k} - \alpha_s^* \alpha^{*k}) \\ &\quad - \frac{1}{6}\tilde{g}_{is}(\nabla_j \alpha^{*k} + \nabla_j \alpha^k - \frac{1}{3}\alpha_j \alpha^{*k} - \frac{1}{3}\alpha_j^* \alpha^{*k}) \\ &\quad + \frac{1}{6}\tilde{g}_{ij}(\nabla_s \alpha^{*k} + \nabla_s \alpha^k - \frac{1}{3}\alpha_s \alpha^{*k} - \frac{1}{3}\alpha_s^* \alpha^{*k}).\end{aligned}$$

Contracting the indices $k = s$ in the above equality, and taking into account (4.13), (4.15), and the expression for the components of the Ricci tensor $\rho_{ij} = R_{ijs}^s$, we obtain

$$\tilde{\rho}_{ij} = \rho_{ij} + \frac{1}{3}g_{ij}\nabla_s \alpha^{*s} + \frac{1}{6}\tilde{g}_{ij}(\nabla_s \alpha^s + \nabla_s \alpha^{*s}). \quad (4.16)$$

Let us denote the associated quantities

$$\tau^* = \tilde{g}^{ij}\rho(e_i, e_j), \quad \tilde{\tau}^* = g^{ij}\tilde{\rho}(e_i, e_j). \quad (4.17)$$

From (2.4), (4.13), (4.16), and (4.17), we get

$$\tilde{\rho}_{ij} = \rho_{ij} + \frac{1}{3}(\tilde{\tau}^* - \tau)g_{ij} + \frac{1}{6}(2\tilde{\tau} - 2\tau^* + \tilde{\tau}^* - \tau)\tilde{g}_{ij}.$$

Consequently, we establish the following result.

Theorem 4.2. *Let (M, g, ϕ) be equipped with the associated metric (4.1). Then, the Ricci tensors ρ and $\tilde{\rho}$, the scalar curvatures τ and $\tilde{\tau}$, and the associated quantities τ^* and $\tilde{\tau}^*$ satisfy the identity*

$$\tilde{\rho}(x, y) = \rho(x, y) + \frac{1}{3}(\tilde{\tau}^* - \tau)g(x, y) + \frac{1}{6}(2\tilde{\tau} - 2\tau^* + \tilde{\tau}^* - \tau)\tilde{g}(x, y). \quad (4.18)$$

Further we will use the following definition, given for another Riemannian manifolds, for example, in [2], [12], [20] and [25].

Definition 4.3. A Riemannian manifold (M, g) is almost Einstein, if the metrics g and $\tilde{g}$ satisfy

$$\rho(x, y) = \alpha g(x, y) + \beta \tilde{g}(x, y), \quad (4.19)$$

where α and β are smooth functions on M .

Proposition 4.4. *If the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ is locally flat, then (M, g, ϕ) is an almost Einstein manifold and its Ricci tensor ρ is given by*

$$\rho(x, y) = \frac{\tau}{3}g(x, y) + \frac{2\tau^* + \tau}{6}\tilde{g}(x, y). \quad (4.20)$$

Proof. If $\tilde{\nabla}$ on (M, g, ϕ) is locally flat, then $\tilde{R} = 0$. From (4.13) and (4.17) it follows that $\tilde{\rho} = 0$ and $\tilde{\tau} = \tau^* = 0$. Thus, Equality (4.18) yields (4.20), and a comparison with (4.19) clearly shows that (M, g, ϕ) is an almost Einstein manifold. $\square$

Remark 4.5. A manifold (M, g, ϕ) is Einstein if $\tau^* = -\frac{1}{2}\tau$. In this case the Ricci tensor has the form

$$\rho(x, y) = \frac{\tau}{3}g(x, y). \quad (4.21)$$

Proposition 4.6. *If the Levi-Civita connection ∇ of g is locally flat, then $(M, \tilde{g}, \phi)$ is an almost Einstein manifolds and its Ricci tensor $\tilde{\rho}$ is*

$$\tilde{\rho}(x, y) = \frac{\tilde{\tau}^*}{3}g(x, y) + \frac{\tilde{\tau}^* + 2\tilde{\tau}}{6}\tilde{g}(x, y). \quad (4.22)$$

The proof is similar to the proof of Proposition 4.4.

This representation of the Ricci tensors enables us to find an explicit expression for the curvature tensors in terms of the metrics and scalar curvatures.

5. CURVATURE PROPERTIES OF (M, g, ϕ)

In this section, we investigate several curvature properties of the manifold (M, g, ϕ) , including the sectional curvatures of special 2-planes in the tangent space $T_p M$, the scalar curvature, and the Ricci curvature in the direction of a non-zero vector. To this end, we first introduce the concept of a ϕ -basis for $T_p M$ spanned by a non-zero vector x and its images under ϕ , and establish the existence of an orthogonal ϕ -basis.

5.1. An orthogonal ϕ -basis in the tangent space. Here, we consider vector fields in a neighborhood of an arbitrary point p of (M, g, ϕ) . Naturally, the results obtained also carry over to the corresponding vectors in $T_p M$.

Lemma 5.1. *For the lengths of the vectors x , ϕx , and $\phi^2 x$, the following identities hold:*

$$\|x\| = \|\phi x\| = \|\phi^2 x\|.$$

Proof. Due to (2.8) and the definition of a length of a vector $\|x\| = \sqrt{g(x, x)}$, we get $g(x, x) = g(\phi x, \phi x) = g(\phi^2 x, \phi^2 x)$; i.e., the lengths are equal. $\square$

Theorem 5.2. *For an arbitrary non-zero vector x and its images ϕx and $\phi^2 x$, the angles between these vectors satisfy the equalities $\theta = \varphi = \psi$, where*

$$\theta = \angle(x, \phi x), \quad \varphi = \angle(\phi x, \phi^2 x), \quad \psi = \angle(\phi^2 x, x). \quad (5.1)$$

Proof. Let the vector $x = (x^1, x^2, x^3)$ be non-zero. Taking into account (5.1), we obtain

$$\cos \theta = \frac{g(x, \phi x)}{g(x, x)}, \quad \cos \varphi = \frac{g(\phi x, \phi^2 x)}{g(x, x)}, \quad \cos \psi = \frac{g(\phi^2 x, x)}{g(x, x)}. \quad (5.2)$$

Identities (2.7) and (2.8) imply $g(x, \phi x) = g(\phi x, \phi^2 x) = g(\phi^2 x, x)$. We apply them to (5.2) and obtain $\cos \theta = \cos \varphi = \cos \psi$. Since the angles are in the interval $[0, \pi]$, we get $\theta = \varphi = \psi$. $\square$

Remark 5.3. Hereafter, we will use the notation θ for each of these three angles.

Theorem 5.4. *The angle θ , determined by an arbitrary non-zero vector x and its images under ϕ , satisfies the inequalities*

$$0 \leq \theta \leq \frac{2\pi}{3}. \quad (5.3)$$

Proof. We consider a vector $x = (x^1, x^2, x^3) \neq o$. The images of x under (2.6) are $\phi x = (x^3, -x^1, -x^2)$ and $\phi^2 x = (-x^2, -x^3, x^1)$. From (2.6) and (2.9), we get

$$g(x, x) = A((x^1)^2 + (x^2)^2 + (x^3)^2) + 2B(x^1 x^2 - x^1 x^3 + x^2 x^3), \quad (5.4)$$

$$g(x, \phi x) = -B((x^1)^2 + (x^2)^2 + (x^3)^2) + (B - A)(x^1 x^2 - x^1 x^3 + x^2 x^3).$$

Substituting (5.4) into (5.2), we find

$$\cos \theta = \frac{-B((x^1)^2 + (x^2)^2 + (x^3)^2) + (B - A)(x^1 x^2 - x^1 x^3 + x^2 x^3)}{A((x^1)^2 + (x^2)^2 + (x^3)^2) + 2B(x^1 x^2 - x^1 x^3 + x^2 x^3)}, \quad (5.5)$$

where $A((x^1)^2 + (x^2)^2 + (x^3)^2) + 2B(x^1 x^2 - x^1 x^3 + x^2 x^3) > 0$.

We suppose that $\theta > \frac{2\pi}{3}$, i.e., $\cos \theta < -\frac{1}{2}$. Thus, from (5.5), we get

$$(A - 2B)(x^1 - x^2 + x^3)^2 < 0.$$

By virtue of condition (2.10), the above inequality can not be satisfied; hence, the assumption $\theta > \frac{2\pi}{3}$ is false. Consequently, the conditions in (5.3) hold. $\square$

As an immediate consequence of Theorem 5.2 and Theorem 5.4, we obtain the following result:

Corollary 5.5. *If $\theta = 0$, then $x = \phi x = \phi^2 x$. If $\theta = \frac{2\pi}{3}$, then the vectors x , ϕx , and $\phi^2 x$ are coplanar.*

Definition 5.6. If the vectors $\{x, \phi x, \phi^2 x\}$ form a basis for the tangent space $T_p M$, it is called a ϕ -basis. In this case, the vector x is said to induce a ϕ -basis for $T_p M$.

Theorem 5.7. *Every tangent space $T_p M$ admits infinitely many ϕ -bases.*

Proof. Let $x = (x^1, x^2, x^3)$ be an arbitrary vector in $T_p M$. Bearing in mind that $\phi x = (x^3, -x^1, -x^2)$ and $\phi^2 x = (-x^2, -x^3, x^1)$, we find that the vectors x , ϕx , and $\phi^2 x$ are linearly independent if and only if

$$\begin{vmatrix} x^1 & x^2 & x^3 \\ x^3 & -x^1 & -x^2 \\ -x^2 & -x^3 & x^1 \end{vmatrix} \neq 0,$$

which implies

$$3x^1 x^2 x^3 \neq -(x^1)^3 + (x^2)^3 - (x^3)^3.$$

Since there are infinitely many triples $x = (x^1, x^2, x^3)$ satisfying this condition, we conclude that there exist infinitely many ϕ -bases for $T_p M$. $\square$

By Theorem 5.4 and Theorem 5.7, we have the following corollary:

Corollary 5.8. *Every tangent space $T_p M$ admits an orthogonal ϕ -basis.*

Example of an orthonormal ϕ -basis

We consider a vector x with coordinates:

$$x^1 = \frac{\sqrt{A+B} + \sqrt{A-3B}}{2\sqrt{A^2 - AB - 2B^2}}, \quad x^2 = 0, \quad x^3 = \frac{\sqrt{A+B} - \sqrt{A-3B}}{2\sqrt{A^2 - AB - 2B^2}}.$$

The following equalities hold:

$$\begin{aligned} (x^1)^2 + (x^2)^2 + (x^3)^2 &= \frac{A-B}{A^2 - AB - 2B^2}, \\ x^1 x^2 - x^1 x^3 + x^2 x^3 &= \frac{-B}{A^2 - AB - 2B^2}. \end{aligned}$$

Substituting these into (5.4) yields

$$g(x, x) = 1, \quad g(x, \phi x) = 0.$$

Thus, from (2.8), we also obtain

$$g(\phi x, \phi x) = g(\phi^2 x, \phi^2 x) = 1, \quad g(x, \phi^2 x) = g(\phi x, \phi^2 x) = 0.$$

By Theorem 5.2, the vector x induces an orthonormal ϕ -basis for $T_p M$.

5.2. Ricci curvatures. By definition, the Ricci curvatures in the direction of a non-zero vector x with respect to g and $\tilde{g}$ are given, respectively, by the expressions:

$$r(x) = \frac{\rho(x, x)}{g(x, x)}, \quad \tilde{r}(x) = \frac{\tilde{\rho}(x, x)}{\tilde{g}(x, x)}. \quad (5.6)$$

With the following theorem, we establish a relationship between these curvatures.

Theorem 5.9. *Let (M, g, ϕ) be equipped with the associated metric (4.1). Suppose that the vector x induces a ϕ -basis, and let θ denote the angle between x and $\phi^2 x$. If $r(x)$ and $\tilde{r}(x)$ are the Ricci curvatures in the direction of x with respect to g and $\tilde{g}$, respectively, then*

$$\tilde{r}(x) = \frac{1}{2 \cos \theta} r(x) + \frac{1}{6 \cos \theta} (\tilde{\tau}^* - \tau) + \frac{1}{6} (2\tilde{\tau} - 2\tau^* + \tilde{\tau}^* - \tau), \quad (5.7)$$

where $\theta \neq \frac{\pi}{2}$.

Proof. From (4.1) and (5.2), we find

$$\tilde{g}(x, x) = 2g(x, \phi x) = 2g(x, x) \cos \theta. \quad (5.8)$$

Using (4.18), we get

$$\frac{\tilde{\rho}(x, x)}{\tilde{g}(x, x)} = \frac{\rho(x, x)}{g(x, x)} + \frac{1}{3} (\tilde{\tau}^* - \tau) \frac{g(x, x)}{g(x, x)} + \frac{1}{6} (2\tilde{\tau} - 2\tau^* + \tilde{\tau}^* - \tau).$$

Then, applying (5.6) to both $r(x)$ and $\tilde{r}(x)$, and taking into account (5.8), we obtain (5.7). $\square$

Next, we consider cases for the Ricci curvatures where one of the two connections, ∇ or $\tilde{\nabla}$, is locally flat.

Proposition 5.10. *Let the Levi-Civita connection $\tilde{\nabla}$ on (M, g, ϕ) be locally flat. If a vector x induces a ϕ -basis, then the Ricci curvatures in the direction of the basis vectors satisfy*

$$r(x) = r(\phi x) = r(\phi^2 x) = \frac{\tau}{3} + \frac{1}{3} (2\tau^* + \tau) \cos \theta, \quad (5.9)$$

where $\theta = \angle(x, \phi x)$.

Proof. Since ρ satisfies (4.20), and bearing in mind (2.8), (5.6), and (5.8), we find (5.9). $\square$

The next two statements can also be verified directly.

Corollary 5.11. *If a vector x induces an orthogonal ϕ -basis, then the Ricci curvatures in the direction of the basis vectors are given by:*

$$r(x) = r(\phi x) = r(\phi^2 x) = \frac{\tau}{3}.$$

Corollary 5.12. *If (M, g, ϕ) is an Einstein manifold and a vector x induces a ϕ -basis, then the Ricci curvatures in the direction of the basis vectors are:*

$$r(x) = r(\phi x) = r(\phi^2 x) = \frac{\tau}{3}.$$

Proof. The proof follows directly by substituting $\tau^* = -\frac{1}{2}\tau$ into (5.9). We note that the Ricci tensor ρ satisfies (4.21). $\square$

Proposition 5.13. *Let the Levi-Civita connection ∇ on $(M, \tilde{g}, \phi)$ be locally flat. If a vector x induces a ϕ -basis, then the Ricci curvatures in the direction of the basis vectors are:*

$$\tilde{r}(x) = \tilde{r}(\phi x) = \tilde{r}(\phi^2 x) = \frac{\tilde{\tau}^*}{6 \cos \theta} + \frac{1}{6}(2\tilde{\tau} + \tilde{\tau}^*), \quad \theta \neq \frac{\pi}{2}.$$

The proof follows directly from (4.2), (4.22), (5.6), and (5.8).

5.3. Curvature tensors. In the following subsection, we express the curvature tensor R of an almost Einstein manifold (M, g, ϕ) in terms of the metrics g and $\tilde{g}$. Analogously, we derive the expression for the curvature tensor $\tilde{R}$ in terms of these metrics.

Theorem 5.14. *Let the Ricci tensor ρ on (M, g, ϕ) be determined by (4.20). Then, the curvature tensor R takes the form*

$$R = \frac{\tau}{6}\pi_1 + \frac{\tau + 2\tau^*}{6}\pi_2, \quad (5.10)$$

where

$$\begin{aligned} \pi_1(x, y, z, u) &= g(y, z)g(x, u) - g(x, z)g(y, u), \\ \pi_2(x, y, z, u) &= g(y, z)\tilde{g}(x, u) + g(x, u)\tilde{g}(y, z) - g(x, z)\tilde{g}(y, u) \\ &\quad - g(y, u)\tilde{g}(x, z). \end{aligned} \quad (5.11)$$

Proof. It is known that the curvature tensor R of a 3-dimensional Riemannian manifold is completely determined by the Ricci tensor ρ (see [15]):

$$R(x, y, z, u) = -g(x, z)\rho(y, u) - g(y, u)\rho(x, z) + g(y, z)\rho(x, u) + g(x, u)\rho(y, z) + \frac{\tau}{2}(g(x, z)g(y, u) - g(y, z)g(x, u)).$$

Substituting (4.20) into the above identity and using the notations introduced in (5.11), we obtain (5.10). $\square$

Since, an Einstein manifold (M, g, ϕ) satisfies $\tau^* = -\frac{\tau}{2}$, relation (5.10) yields the following result:

Corollary 5.15. *If the Ricci tensor ρ on (M, g, ϕ) is defined by (4.21), then the curvature tensor R is given by*

$$R = \frac{\tau}{6}\pi_1, \quad (5.12)$$

where $\pi_1(x, y, z, u) = g(y, z)g(x, u) - g(x, z)g(y, u)$.

The following theorem provides the representation of the curvature tensor $\tilde{R}$ in terms of the metrics g and $\tilde{g}$.

Theorem 5.16. *Let the Ricci tensor $\tilde{\rho}$ on $(M, \tilde{g}, \phi)$ be defined by (4.22). Then, the associated curvature tensor $\tilde{R}$ takes the form*

$$\tilde{R} = \frac{\tilde{\tau} + 2\tilde{\tau}^*}{6}\tilde{\pi}_1 + \frac{\tilde{\tau}^*}{3}\pi_2,$$

where

$$\begin{aligned} \tilde{\pi}_1(x, y, z, u) &= \tilde{g}(y, z)\tilde{g}(x, u) - \tilde{g}(x, z)\tilde{g}(y, u), \\ \pi_2(x, y, z, u) &= g(y, z)\tilde{g}(x, u) + g(x, u)\tilde{g}(y, z) \\ &\quad - g(x, z)\tilde{g}(y, u) - g(y, u)\tilde{g}(x, z). \end{aligned}$$

Proof. Curvature tensor $\tilde{R}$ is completely determined by the Ricci tensor $\tilde{\rho}$. Then, from (4.22), we obtain the statement. $\square$

5.4. Sectional curvatures. Using the representation of the curvature tensor R obtained in the previous subsection, we derive expressions for the sectional curvatures of special 2-planes in the tangent space $T_p M$ of (M, g, ϕ) .

Let a vector x induce a ϕ -basis. Then we have three 2-planes in $T_p M$ generated by pairs from the basis $\{x, \phi x, \phi^2 x\}$. In the following statement, we find a representation of the sectional curvatures of these 2-planes by the angle θ and the scalar quantities τ and τ^* .

Theorem 5.17. *Let the curvature tensor R on (M, g, ϕ) be expressed by (5.10), and suppose that a vector x induces a ϕ -basis. Then, the sectional curvatures of the 2-planes determined by the basis vectors satisfy*

$$\begin{aligned} k(x, \phi x) &= k(\phi x, \phi^2 x) = k(x, \phi^2 x) \\ &= \frac{1}{6(1 + \cos \theta)} (\tau(3 \cos \theta + 1) + 4\tau^* \cos \theta), \end{aligned} \quad (5.13)$$

where $\theta = \angle(x, \phi x)$.

Proof. The conditions (2.8), (5.1) and (5.2) imply

$$g(x, \phi x) = g(\phi x, \phi^2 x) = g(x, \phi^2 x) = g(x, x) \cos \theta. \quad (5.14)$$

Thus, from (4.1), (4.2), and (5.14) we find

$$\tilde{g}(x, x) = \tilde{g}(\phi x, \phi x) = \tilde{g}(\phi^2 x, \phi^2 x) = 2g(x, x) \cos \theta, \quad (5.15)$$

as well as

$$\tilde{g}(x, \phi x) = \tilde{g}(x, \phi^2 x) = \tilde{g}(\phi x, \phi^2 x) = g(x, x)(1 + \cos \theta). \quad (5.16)$$

Next, setting $y = \phi x$ into (5.11) and applying (5.14), (5.15), and (5.16), we successively obtain:

$$\begin{aligned} \pi_1(x, \phi x, x, \phi x) &= (\cos^2 \theta - 1)g^2(x, x), \\ \pi_2(x, \phi x, x, \phi x) &= 2(\cos^2 \theta - \cos \theta)g^2(x, x). \end{aligned}$$

Substituting these expressions into (5.10) and using (2.5), we obtain the sectional curvature formula:

$$k(x, \phi x) = \frac{1}{6(1 + \cos \theta)} (\tau(3 \cos \theta + 1) + 4\tau^* \cos \theta).$$

Since the identities

$$\begin{aligned} \pi_1(x, \phi x, x, \phi x) &= \pi_1(x, \phi^2 x, x, \phi^2 x) = \pi_1(\phi x, \phi^2 x, \phi x, \phi^2 x), \\ \pi_2(x, \phi x, x, \phi x) &= \pi_2(x, \phi^2 x, x, \phi^2 x) = \pi_2(\phi x, \phi^2 x, \phi x, \phi^2 x), \end{aligned}$$

hold and (5.14) is satisfied, we conclude that the relations in (5.13) are valid. $\square$

The following corollaries can be verified by direct calculations.

Corollary 5.18. *Let a vector x induce an orthogonal ϕ -basis. Then the sectional curvatures of the 2-planes defined by the basis vectors are:*

$$k(x, \phi x) = k(\phi x, \phi^2 x) = k(x, \phi^2 x) = \frac{\tau}{6}.$$

In particular, for an Einstein manifold, the theorem reduces to the following result:

Corollary 5.19. *If the curvature tensor R on (M, g, ϕ) is given by (5.12), then the sectional curvatures of the 2-planes satisfy*

$$k(x, \phi x) = k(\phi x, \phi^2 x) = k(x, \phi^2 x) = \frac{\tau}{6}.$$

6. EQUATION OF A SPHERE WITH RESPECT TO THE ASSOCIATED METRIC

In this section, we consider the equation of a sphere defined with respect to the indefinite metric $\tilde{g}$. Relative to the Riemannian metric g , this equation transforms into the equation of a quadric surface.

The associated metric $\tilde{g}$ on (M, g, ϕ) is defined by (4.1). Next, using (2.7) and (2.8), we obtain the following relationship between g and $\tilde{g}$:

$$\tilde{g}(u, u) = 2g(u, \phi u), \quad \tilde{g}(u, \phi u) = \tilde{g}(u, \phi^2 u) = g(u, u) + g(u, \phi u). \quad (6.1)$$

Here, u is an arbitrary vector, and ϕu and $\phi^2 u$ are its images under ϕ .

According to physical terminology, we state the following definition for $\tilde{g}$:

Definition 6.1. Let $\tilde{g}$ be the associated metric on (M, g, ϕ) . If a vector u satisfies $\tilde{g}(u, u) > 0$ (resp. $\tilde{g}(u, u) < 0$), then u is a spacelike vector (or a timelike vector, respectively). If a non-zero vector u satisfies $\tilde{g}(u, u) = 0$, then u is a null vector.

Taking into account (5.15) and (6.1), we prove the following statement:

Theorem 6.2. *Let $\tilde{g}$ be the associated metric on (M, g, ϕ) , and let θ be the angle between u and ϕu with respect to g . Then the following assertions hold:*

- (i) *a vector u is spacelike if and only if $\theta \in [0, \frac{\pi}{2})$;*
- (ii) *a vector u is null if and only if $\theta = \frac{\pi}{2}$;*
- (iii) *a vector u is timelike if and only if $\theta \in (\frac{\pi}{2}, \frac{2\pi}{3}]$.*

Clearly, taking (5.15) into account, we have the following proposition.

Proposition 6.3. *If u is a spacelike (null or timelike) vector, then ϕu and $\phi^2 u$ are spacelike (null or timelike) vectors, respectively.*

Let $\{u, \phi u, \phi^2 u\}$ be an orthonormal ϕ -basis for $T_p M$, i.e., $\theta = \frac{\pi}{2}$. Let p_{XYZ} be a coordinate system, such that the vectors u , ϕu , and $\phi^2 u$ lie along the axes p_X , p_Y and p_Z , respectively. Then p_{XYZ} is an orthonormal coordinate system. If $N(X, Y, Z)$ is an arbitrary point with a position vector v , then v is expressed by

$$v = Xu + Y\phi u + Z\phi^2 u. \quad (6.2)$$

The equation of a central sphere s with a radius r , with respect to $\tilde{g}$, is given by

$$s : \quad \tilde{g}(v, v) = r^2. \quad (6.3)$$

Since $\tilde{g}$ is an indefinite metric, it is possible for the square of r to satisfy $r^2 > 0$, $r^2 = 0$, or $r^2 < 0$.

Substituting (6.2) into (6.3), we find

$$(X^2 + Y^2 + Z^2)\tilde{g}(u, u) + (2XY + 2YZ)\tilde{g}(u, \phi u) + 2XZ\tilde{g}(u, \phi^2 u) = r^2.$$

Taking into account Equalities (5.15), (5.16), (6.1), and $\theta = \frac{\pi}{2}$, we obtain the equation of s :

$$2(XY + YZ + XZ) = r^2. \quad (6.4)$$

We rotate the coordinate system p_{xyz} into $p_{x'y'z'}$ using the transformation

$$\begin{aligned} X &= \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z', \\ Y &= \frac{\sqrt{2}}{\sqrt{3}}y' + \frac{1}{\sqrt{3}}z', \\ Z &= -\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z'. \end{aligned} \quad (6.5)$$

Then, Equation (6.4) reduces to the following:

$$x'^2 + y'^2 - 2z'^2 = -r^2. \quad (6.6)$$

- If $r^2 = 0$, Equation (6.6) defines a cone:

$$s_0 : x'^2 + y'^2 - 2z'^2 = 0. \quad (6.7)$$

- If $r^2 > 0$, Equation (6.6) yields the equation of a hyperboloid of two sheets:

$$s_1 : x'^2 + y'^2 - 2z'^2 = -a^2, \quad a^2 = r^2 > 0. \quad (6.8)$$

- If $r^2 < 0$, Equation (6.6) reduces to the equation of a hyperboloid of one sheet:

$$s_2 : x'^2 + y'^2 - 2z'^2 = a^2, \quad a^2 = -r^2 > 0. \quad (6.9)$$

Consequently, we establish the following result:

Theorem 6.4. *Let $\tilde{g}$ be the associated metric on (M, g, ϕ) , and let the vectors u , ϕu and $\phi^2 u$ form an orthonormal ϕ -basis of $T_p M$. Suppose that p_{XYZ} is a coordinate system such that $u \in p_X$, $\phi u \in p_Y$ and $\phi^2 u \in p_Z$. Then the sphere s determined by (6.3) is given by Equation (6.6) with respect to $p_{x'y'z'}$, which is obtained via the rotation (6.5) of p_{XYZ} . In particular:*

- (i) if $r^2 = 0$, then s is a cone s_0 defined by (6.7);

- (ii) if $r^2 > 0$, then s is a hyperboloid of two sheets s_1 defined by (6.8);
- (iii) if $r^2 < 0$, then s is a hyperboloid of one sheet s_2 defined by (6.9).

Remark 6.5. We shall say that the surfaces s_0 , s_1 , and s_2 are induced by the sphere s .

Corollary 6.6. *If the surfaces s_0 , s_1 , and s_2 are induced by the sphere s , then:*

- (i) every point of s_0 has an null position vector;
- (ii) every point of s_1 has a spacelike position vector;
- (iii) Every point of s_2 has a timelike position vector.

Proof. This statement follows from Definition 6.1, Equalities (6.1) and (6.3), the conditions for r and the definitions of the three surfaces s_0 , s_1 and s_2 . $\square$

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**Върху геометрията на тримерно риманово многообразие с
косоциркулантна структура и асоциирана метрика**

Кристияна Темнилова, Ива Докузова

Резюме. Разгледано е тримерно риманово многообразие M с метричен тензор g и допълнителна тензорна структура ϕ от тип $(1, 1)$. Локалните компоненти на тези два тензора образуват косоциркулантни матрици. Третата степен на ϕ е равна на идентитета и ϕ е съвместима с g .

Многообразието (M, g, ϕ) е снабдено с асоциирана неопределителна метрика $\tilde{g}$, определена от g и ϕ . Установено е, че ако свързаността на Леви-Чевита за $\tilde{g}$ е локално плоска, то (M, g, ϕ) е почти айнщайново многообразие.

За почти айнщайново многообразие (M, g, ϕ) са изчислени кривините на Ричи по направление на базисните вектори за произволна ϕ -база от допирателното пространство $T_p M$. Намерени са изрази за тензорите на кривина R и $\tilde{R}$ чрез метриките g и $\tilde{g}$ и скаларните кривини. Получени са равенства между секционните кривини на специални площадки, определени от базисните вектори на произволна ϕ -база.

Разгледано е уравнение на сфера спрямо $\tilde{g}$. Тъй като $\tilde{g}$ е неопределителна метрика, уравнението на сферата спрямо g се прообразува в уравнения на три вида квадратични повърхнини, вложени в евклидовото пространство. Това са конус, прост хиперболоид и двоен хиперболоид.