

CONTEXT-FREE FIXED POINTS AND COMPLETE CLASSIFICATION OF ORBITS IN PICARD ITERATION FOR GUARDED POWER LANGUAGE OPERATORS

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ABSTRACT. We study the language-theoretic structure of fixed points and finite Picard iterates for guarded q -power language operators. For the general operator, we prove that a context-free seeded language determines a unique context-free fixed point and give an effective construction of a context-free grammar generating this fixed point, independently of the initial language. We then consider a marked single-guard special case in which two new symbols separate the recursive contribution from the contribution of the seed. In this setting, the initial language can be traced and recovered exactly from every finite Picard iterate by means of a regular slice and fixed-word quotients. This yields injectivity of the finite-time maps and an abstract finite-time class-preservation principle. As a consequence, we obtain a classification of the finite Picard iterates according to the exact position of the initial language in the Chomsky hierarchy. Thus the exact language-theoretic complexity may persist at every finite stage, while all Picard orbits converge to the same context-free fixed point.

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1. INTRODUCTION

Formal languages, grammars, and automata admit several natural fixed-point interpretations. In particular, context-free grammars can be represented by finite systems of language equations, with the languages generated by the nonterminals forming the least solution of the corresponding system. Fixed-point principles in formal language theory have been studied for decades, both from the general viewpoint of language equations and through order-theoretic and metric methods [1, 2, 3]. More recent investigations have substantially extended the theory of language equations, including equations involving concatenation and Boolean operations, their expressive power, uniqueness of solutions, and associated decision and computability problems [4, 5].

A complementary viewpoint is obtained by equipping the space of formal languages with a metric determined by the shortest word on which two languages disagree. This topology goes back to the classical work on metric spaces of languages and has subsequently been used in the study of topological properties of language classes [6, 7]. Within this framework, guarded language operators were recently introduced as contractive transformations whose recursive part delays observable differences by a prescribed word length [8]. The framework has since developed in several directions. Positive–negative guarded systems have been studied by combining the same ultrametric geometry with a suitable product order and weak monotonicity, leading to ordered fixed-point results for systems containing both positive and complemented recursive dependencies [9]. Finite-observation conditional guarded operators provide another extension, in which the operator may depend on finitely observable properties of the input language and fixed-point behaviour can be governed by Kannan-type conditions beyond ordinary Banach contractivity [10]. A further fixed-power extension yields guarded operators with a prescribed positive language power together with explicit contraction estimates, uniqueness of the fixed point, and quantitative Picard convergence [11].

The present paper develops two complementary aspects of the dynamics of guarded q -power language operators. We first study the properties of the fixed point for the general operator. For a context-free seed, we prove that the unique fixed point is itself context-free and also give an explicit effective construction of a context-free grammar generating it. The construction is independent of the initial language and shows directly that the context-free nature of the fixed point is a structural consequence of the fixed-point equation, rather than merely a consequence of invariance of the context-free class.

We then turn from the fixed point to the finite Picard iterates. A complete classification cannot be expected for the general operator without additional

separation assumptions, since the seed language may dominate the recursive contribution and destroy all information about the initial language after finitely many steps. We therefore consider a marked single-guard special case of the general q -power operator, in which two new symbols for each guard distinguish the recursive part from the seed. For this dynamics, we prove that the initial language can be isolated by a regular slice and recovered exactly from every finite Picard iterate. This leads to injectivity of the finite-time maps, a general class-preservation principle, and a classification of the finite iterates according to the exact position of the initial language in the Chomsky hierarchy. In particular, the exact language-theoretic class of the initial state may persist at every finite stage even though all Picard orbits converge to the same context-free fixed point.

The paper is organized as follows. For the reader's convenience, Section 2 includes an overview of some standard definitions, basic properties and previously obtained results, used later in the paper. In Section 3 we study the language-theoretic structure of the unique fixed point of the guarded q -power language operator introduced in [11]. We prove that if the seeded language is context-free, then independent from the initial language, the unique fixed point of this operator is context-free. In Section 4 as our second main result the relation between the finite Picard orbits is studied and a full language classification of the finite Picard iterates and their fixed points for the marked single guarded special case of the guarded q -power language operator is obtained. In Section 5 some examples, illustrating the obtained theoretical results are presented.

2. PRELIMINARIES

This section collects the standard definitions, previously established results, and closure properties used in the sequel. We recall the basic language-theoretic notation, the length-based ultrametric on the space of formal languages, the previously established contraction theorem for guarded q -power operators, the Chomsky hierarchy and the relevant closure properties of its language classes, and the classical connection between context-free grammars and language equations. No substantive result in this section is claimed as new.

Throughout the paper, Σ denotes a fixed finite nonempty alphabet, Σ^* denotes the set of all finite words over Σ , including the empty word ε , and $\mathcal{L} = \mathcal{P}(\Sigma^*)$ denotes the set of all formal languages over Σ . For $w \in \Sigma^*$, its length is denoted by $|w|$.

Definition 2.1 ([12, 13]). A formal language over Σ is any subset $L \subseteq \Sigma^*$. Its characteristic function is denoted by $\mathbf{1}_L : \Sigma^* \rightarrow \{0, 1\}$, where $\mathbf{1}_L(w) = 1$ if $w \in L$ and $\mathbf{1}_L(w) = 0$ otherwise.

For languages $A, B \subseteq \Sigma^*$, their concatenation is $AB = \{xy : x \in A, y \in B\}$. For $q \geq 1$, A^q denotes the q -fold concatenation of A with itself. If $u, v \in \Sigma^*$ and $L \subseteq \Sigma^*$, we write $uLv = \{uvw : w \in L\}$. For a fixed word $u \in \Sigma^*$, the left quotient of L by u is $u^{-1}L = \{w \in \Sigma^* : uw \in L\}$, and for $v \in \Sigma^*$, the right quotient is $Lv^{-1} = \{w \in \Sigma^* : wv \in L\}$. These conventions are standard in formal language theory [12, 13, 14].

Definition 2.2 ([12, 13, 14]). For a language $L \subseteq \Sigma^*$, its Kleene star is the language $L^* = \bigcup_{n=0}^{\infty} L^n$, where $L^0 = \{\varepsilon\}$. Thus L^* consists of all words obtained by concatenating zero or more words from L , with the concatenation of zero words understood to be ε . In particular, $\varepsilon \in L^*$ for every language L , and $*$ = $\{\varepsilon\}$.

The metric used throughout the paper is the classical first-disagreement metric on formal languages considered in the Bodnarchuk–Vianu framework and in later works on the topology of language classes [6, 7].

Definition 2.3 ([6, 7]). For $L, M \in \mathcal{L}$, put $\nu(L, L) = \infty$, while for $L \neq M$ define $\nu(L, M) = \min\{|w| : \mathbf{1}_L(w) \neq \mathbf{1}_M(w)\}$. With the convention $2^{-\infty} = 0$, define $d(L, M) = 2^{-\nu(L, M)}$.

Thus $\nu(L, M)$ is the length of the shortest word distinguishing L from M . The distance therefore compares languages according to the shortest finite observation on which they disagree [6, 7].

Lemma 2.4 ([6, 7]). Let $L, M \in \mathcal{L}$ and $N \geq 0$. Then $d(L, M) \leq 2^{-N}$ if and only if L and M agree on every word $w \in \Sigma^*$ with $|w| < N$.

Definition 2.5 ([15]). Let X be a nonempty set. A function $\rho : X \times X \rightarrow [0, \infty)$ is an ultrametric if $\rho(x, y) = 0$ if and only if $x = y$, $\rho(x, y) = \rho(y, x)$, and $\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}$ for all $x, y, z \in X$.

Theorem 2.6 ([6, 7]). The function d is an ultrametric on $\mathcal{L}$, and the ultrametric space $(\mathcal{L}, d)$ is complete. The induced topology is the Cantor-type topology obtained by identifying each language with its characteristic function in $\{0, 1\}^{\Sigma^*}$.

Theorem 2.7 ([7]). The family of the finite languages is dense in $(\mathcal{L}, d)$.

We shall use the classical Banach contraction principle in its standard form.

Theorem 2.8 ([16]). Let (X, ρ) be a nonempty complete metric space and let $T : X \rightarrow X$ satisfy $\rho(Tx, Ty) \leq c\rho(x, y)$ for all $x, y \in X$, where $0 \leq c < 1$. Then T has a unique fixed point $x^{(*)}$. For every initial point $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ converges to $x^{(*)}$, and $\rho(x_n, x^{(*)}) \leq c^n \rho(x_0, x^{(*)})$ for every $n \geq 0$.

The guarded language operators and their contraction theory were introduced in [8]. The fixed-power generalization considered in the present paper was established in [11].

Definition 2.9 ([11]). Let $q \geq 1$ be an integer, let $S \subseteq \Sigma^*$ be a seed language, and let $G = \{(u_r, v_r) : r = 1, \dots, p\} \subseteq \Sigma^* \times \Sigma^*$ be a finite nonempty family of guards. The corresponding guarded q -power language operator $T_q : \mathcal{L} \rightarrow \mathcal{L}$ is defined by

$$T_q(L) = S \cup \left(\bigcup_{r=1}^p u_r L v_r \right)^q.$$

Its minimum guard length is $m = \min_{1 \leq r \leq p} (|u_r| + |v_r|)$.

For $q = 1$, Definition 2.9 reduces to the guarded language operator studied in [8].

Theorem 2.10 ([11]). *Let T_q be the operator from Definition 2.9, and assume that $m \geq 1$. Then $d(T_q(L), T_q(M)) \leq 2^{-qm} d(L, M)$ for all $L, M \in \mathcal{L}$. Consequently, T_q has a unique fixed point $L_q^{(*)} \in \mathcal{L}$. For every initial language $L^{(0)} \in \mathcal{L}$, the Picard iteration $L^{(n+1)} = T_q(L^{(n)})$ converges to $L_q^{(*)}$, and $d(L^{(n)}, L_q^{(*)}) \leq 2^{-qmn} d(L^{(0)}, L_q^{(*)})$ for every $n \geq 0$.*

Remark 2.11. To avoid confusion, we denote the fixed point of T_q by $L_q^{(*)}$ throughout the paper, whereas L^* denotes the Kleene star of a language L .

We next recall the standard formal language classes of the Chomsky hierarchy and the closure properties that will be used later.

Definition 2.12 ([13, 14]). A deterministic finite automaton over Σ is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite nonempty set of states, $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of accepting states. A language is regular if it is recognized by some deterministic finite automaton.

Definition 2.13 ([12, 14]). A formal grammar is a tuple $\mathcal{G} = (V, \Sigma, P, S_0)$, where V is a finite set of nonterminal symbols, $V \cap \Sigma = \emptyset$, $S_0 \in V$ is the start symbol, and P is a finite set of productions $\alpha \rightarrow \beta$, where $\alpha \in (V \cup \Sigma)^* V (V \cup \Sigma)^*$ and $\beta \in (V \cup \Sigma)^*$.

The classes of the Chomsky hierarchy are defined as follows.

- (i) A grammar is *unrestricted* (Type 0) if no further restriction is imposed on its productions. The languages generated by such grammars form the class $\mathcal{L}_{\text{un}}$. Equivalently, $\mathcal{L}_{\text{un}}$ is the class of recursively enumerable languages.

- (ii) A grammar is *context-sensitive* (Type 1) if every production $\alpha \rightarrow \beta$ satisfies $|\alpha| \leq |\beta|$, with the standard possible exception $S_0 \rightarrow \varepsilon$, provided that S_0 does not occur on the right-hand side of any production. The corresponding class of languages is denoted by $\mathcal{L}_{\text{cs}}$.
- (iii) A grammar is *context-free* (Type 2) if every production has the form

$$A \rightarrow \alpha, \quad A \in V, \quad \alpha \in (V \cup \Sigma)^*.$$

The corresponding class of languages is denoted by $\mathcal{L}_{\text{cf}}$.

- (iv) A grammar is *regular* (Type 3) if it is right-linear; its productions may be taken in the form $A \rightarrow aB$ or $A \rightarrow a$, where $A, B \in V$ and $a \in \Sigma$, with the standard possible exception $S_0 \rightarrow \varepsilon$, provided that S_0 does not occur on the right-hand side of any production. The corresponding class of languages is denoted by $\mathcal{L}_{\text{reg}}$.

Thus the Chomsky hierarchy gives the inclusions

$$\mathcal{L}_{\text{reg}} \subset \mathcal{L}_{\text{cf}} \subset \mathcal{L}_{\text{cs}} \subset \mathcal{L}_{\text{un}} \subset \mathcal{L}.$$

It is well known that for $|\Sigma| > 1$ (as in the case considered by us later) all these inclusions are strict.

For later use, we also denote by $\mathcal{L}_{\text{nrn}} = \mathcal{L} \setminus \mathcal{L}_{\text{un}}$ the class of languages that are not recursively enumerable.

Theorem 2.14 ([12, 14]). *Each of the classes $\mathcal{L}_{\text{reg}}$, $\mathcal{L}_{\text{cf}}$, $\mathcal{L}_{\text{cs}}$, $\mathcal{L}_{\text{un}}$ is closed under finite union and concatenation, under intersection with a regular language, and under left and right quotient by a fixed word.*

Theorem 2.15 ([12, 13, 14]). *The class of context-free languages is closed under finite union, concatenation, Kleene star, homomorphism, inverse homomorphism, intersection with a regular language, and left and right quotient by a regular language. In particular, if $L \in \mathcal{L}_{\text{cf}}$ and $u, v \in \Sigma^*$, then uLv is context-free, and the left and right quotients of L by fixed words are context-free.*

As an immediate consequence of the standard closure properties in Theorem 2.15, the guarded q -power operator from Definition 2.9 preserves context-freeness whenever its seed is context-free.

Corollary 2.16. *If $S \in \mathcal{L}_{\text{cf}}$, then $T_q(\mathcal{L}_{\text{cf}}) \subseteq \mathcal{L}_{\text{cf}}$.*

Lemma 2.17 ([12, 14]). *If the alphabet contains three distinct symbols a, b, c , then the language $\{a^n b^n c^n : n \geq 1\}$ is not context-free.*

Finite languages are regular and therefore context-free [13, 14]. Hence Theorem 2.7 implies that $\mathcal{L}_{\text{cf}}$ is dense in $\mathcal{L}$. It is also standard that over every finite nonempty alphabet there exist languages that are not context-free [12, 14].

Corollary 2.18. *The metric subspace $(\mathcal{L}_{\text{cf}}, d)$ is incomplete.*

Remark 2.19. The preceding corollary is a consequence of the standard density and language-class properties recalled above [7, 12, 14]. Its relevance here is that invariance of the context-free class under an operator does not, by itself, ensure that the limit of a convergent Picard orbit is context-free. The context-free nature of the limiting fixed point therefore requires a separate justification.

The following algebraic interpretation of context-free grammars by systems of language equations is well known.

Theorem 2.20 ([3, 4]). *Every context-free grammar can be represented by a finite system of language equations whose right-hand sides are formed from finite unions and concatenations of variables and terminal words. The languages generated by the nonterminals of the grammar form the least solution of the corresponding system. Conversely, finite systems of this grammar form provide an equivalent algebraic representation of context-free grammars.*

3. CONTEXT-FREE FIXED POINTS OF GUARDED q -POWER LANGUAGE OPERATORS

In this section we study the language-theoretic structure of the unique fixed point of the general guarded q -power language operator from Definition 2.9. Throughout the section, T_q denotes this operator and its minimum guard length is assumed to satisfy $m \geq 1$. By Theorem 2.10, there exists a unique fixed point $L_q^{(*)} \in \mathcal{L}$ and every Picard orbit converges to it independently of the chosen initial language.

The contraction theorem determines existence, uniqueness, and convergence, but it does not determine the language-theoretic nature of the fixed point. In particular, although Corollary 2.16 shows that the context-free class is invariant under T_q , this alone is insufficient to conclude that the limiting language is context-free, since the context-free subspace is incomplete in the inherited ultrametric. Therefore we begin by determining the language class of the unique fixed point directly.

Theorem 3.1. *Let $q \geq 1$ and $m \geq 1$. If the seed language S is context-free, then the unique fixed point $L_q^{(*)}$ of T_q is context-free.*

Proof. Let $\mathcal{G}_S = (V, \Sigma, P, S_0)$ be a context-free grammar generating S . Choose two new symbols $X, B \notin V \cup \Sigma$, regard them as new nonterminals, and define the grammar $\mathcal{G}'_q = (V \cup \{X, B\}, \Sigma, P'_q, X)$, where

$$P'_q = P \cup \{X \rightarrow S_0, X \rightarrow B^q\} \cup \{B \rightarrow u_r X v_r : r = 1, \dots, p\}.$$

Here B^q denotes the word consisting of q consecutive occurrences of the non-terminal B . Since V and P are finite and only finitely many new nonterminals and productions are added, $\mathcal{G}'_q$ is a context-free grammar.

Let K denote the language generated from X . Since the only productions with left-hand side B are $B \rightarrow u_r X v_r$, the language generated from B is exactly $\bigcup_{r=1}^p u_r K v_r$. Hence $X \rightarrow B^q$ generates exactly $(\bigcup_{r=1}^p u_r K v_r)^q$, while $X \rightarrow S_0$ generates S . Therefore

$$K = S \cup \left(\bigcup_{r=1}^p u_r K v_r \right)^q = T_q(K).$$

Thus K is a fixed point of T_q .

According Theorem 2.10, T_q has a unique fixed point $L_q^{(\star)}$ in $\mathcal{L}$. Consequently, $K = L_q^{(\star)}$. Since $\mathcal{G}'_q$ is a context-free grammar generating K , the language $L_q^{(\star)}$ is context-free. $\square$

Proposition 3.2. *Let $q \geq 1$, $m \geq 1$, and suppose that S is generated by a context-free grammar $\mathcal{G}_S = (V, \Sigma, P, S_0)$. A context-free grammar for $L_q^{(\star)}$ can be constructed effectively from $\mathcal{G}_S$, q , and G . The construction introduces exactly two additional nonterminals and $p+2$ additional productions. In particular, it produces a grammar with $|V|+2$ nonterminals and $|P|+p+2$ productions.*

Proof. Use the grammar $\mathcal{G}'_q$ constructed in the proof of Theorem 3.1. Besides the productions of P , it contains the two productions $X \rightarrow S_0$ and $X \rightarrow B^q$, together with one production $B \rightarrow u_r X v_r$ for each $r = 1, \dots, p$. Thus exactly two new nonterminals and $p+2$ new productions are introduced. Theorem 3.1 shows that this grammar generates $L_q^{(\star)}$. $\square$

Remark 3.3. The construction in Proposition 3.2 depends only on a grammar for the seed, the fixed exponent q , and the guard family. It is completely independent of the initial language used in the Picard iteration. In particular, no effective representation of the initial language is required in order to construct a context-free grammar generating the limiting fixed point.

Corollary 3.4. *Let $q \geq 1$, $m \geq 1$, and assume that S is context-free. For every initial language $L^{(0)} \in \mathcal{L}$, the Picard iteration $L^{(n+1)} = T_q(L^{(n)})$ converges to the same context-free fixed point $L_q^{(\star)}$, and*

$$d(L^{(n)}, L_q^{(\star)}) \leq 2^{-qmn} d(L^{(0)}, L_q^{(\star)}) \quad \text{for every } n \geq 0. \quad (3.1)$$

Proof. The convergence and estimate (3.1) follow from Theorem 2.10, while Theorem 3.1 shows that the common limit $L_q^{(\star)}$ is context-free. $\square$

Remark 3.5. No language-theoretic restriction is imposed on $L^{(0)}$ in Corollary 3.4. In particular, the initial language need not be context-free, decidable, or recursively enumerable. It may be an arbitrary element of $\mathcal{P}(\Sigma^*)$, while the unique limit of its Picard orbit is nevertheless context-free whenever the seed is context-free. Thus the language-theoretic complexity of the initial state may affect the finite orbit but does not determine the language class of its unique limit.

Corollary 3.6. *Let $q \geq 1$, $m \geq 1$, and assume that both S and $L^{(0)}$ are context-free. Then every Picard iterate $L^{(n)}$ is context-free and $L^{(n)} \rightarrow L_q^{(*)}$, where $L_q^{(*)}$ is context-free.*

Proof. The assertion follows by induction from Corollary 2.16. The initial language $L^{(0)}$ is context-free by assumption, and whenever $L^{(n)}$ is context-free, Corollary 2.16 gives $L^{(n+1)} = T_q(L^{(n)}) \in \mathcal{L}_{\text{cf}}$. The convergence follows from Corollary 3.4 and Theorem 3.1 gives $L_q^{(*)} \in \mathcal{L}_{\text{cf}}$. $\square$

Remark 3.7. The conclusion $L_q^{(*)} \in \mathcal{L}_{\text{cf}}$ does not follow merely from Corollary 3.6. A convergent sequence of context-free languages may have a non-context-free limit in the complete ambient language space because $(\mathcal{L}_{\text{cf}}, d)$ is incomplete. Theorem 3.1 is therefore a structural statement about the fixed-point equation itself rather than a consequence of invariance of the context-free class.

4. CLASSIFICATION OF LANGUAGE CLASSES OF THE FINITE PICARD ITERATES AND OF THE FIXED POINT FOR A SPECIAL CASE OF THE GUARDED q -POWER OPERATOR

The preceding section determines the language-theoretic nature of the unique fixed point for the general guarded q -power operator. We now turn to a different question: what can be said about the language classes of the finite Picard iterates before the limit fixed point is reached?

For the general operator from Definition 2.9, no classification of the finite iterates solely in terms of the initial language can hold without additional assumptions. Indeed, consider the admissible choice $S = \Sigma^*$. Then, for every language $L \subseteq \Sigma^*$, $T_q(L) = \Sigma^* \cup (\bigcup_{r=1}^p u_r L v_r)^q = \Sigma^*$. Consequently, independently of the initial language $L^{(0)}$, one has $L^{(1)} = \Sigma^*$ and all subsequent Picard iterates coincide with the fixed point Σ^* which is regular. Thus the seed may completely dominate the recursive part of the operator and erase, already after one step, any language-theoretic distinction between different initial languages.

The preceding example shows that the general operator T_q may erase all information about the initial language after a single Picard step. To prevent the seed from masking the recursive contribution, we introduce a marked special case, denoted by T_{qm} . Two new symbols mark the left and right guards and distinguish the recursively generated words from the seed, while both the seed and the initial language remain over the original alphabet. This separation is the basis for the possibility of exact recovery of the initial language from every finite Picard iterate and for the classification established below under the corresponding assumptions on the seed and the initial languages.

Definition 4.1. Let Σ be a finite nonempty alphabet, let $x, y \notin \Sigma$ be two distinct new symbols, and put $\Gamma = \Sigma \cup \{x, y\}$. Let $q \geq 1$ be an integer and let $S \subseteq \Sigma^*$ be a seed language. The marked single-guard q -power language operator $T_{qm} : \mathcal{P}(\Gamma^*) \rightarrow \mathcal{P}(\Gamma^*)$ is defined for $L \subseteq \Gamma^*$ by

$$T_{qm}(L) = S \cup (xLy)^q.$$

Here the letter m in the subscript is a fixed label indicating the marked case and does not denote the minimum guard length.

The operator T_{qm} is the special case of Definition 2.9 over the enlarged alphabet Γ obtained by taking $p = 1$ and the single guard $(u_1, v_1) = (x, y)$. The seed and the initial language are assumed to satisfy $S, L^{(0)} \subseteq \Sigma^* \subset \Gamma^*$.

Throughout the remainder of this section, the ambient alphabet is Γ . Accordingly, we write $\mathcal{L} = \mathcal{P}(\Gamma^*)$, and the symbols $\mathcal{L}_{\text{reg}}$, $\mathcal{L}_{\text{cf}}$, $\mathcal{L}_{\text{cs}}$, $\mathcal{L}_{\text{un}}$ refer to the corresponding language classes over Γ . Languages over the original alphabet Σ are identified with their natural copies in Γ^* . This enlargement of the ambient alphabet does not change their membership in any of the classes of the Chomsky hierarchy. In particular, $\mathcal{L}_{\text{nrn}} = \mathcal{L} \setminus \mathcal{L}_{\text{un}}$ denotes, in this section, the class of non-recursively enumerable languages over Γ .

The same convention applies to the language ultrametric. Throughout this section, d denotes the first-disagreement ultrametric of Definition 2.3 on $\mathcal{P}(\Gamma^*)$, with word lengths taken in Γ^* . Consequently, Theorem 2.6 and the contraction result in Theorem 2.10 apply to the marked operator over the ambient alphabet Γ .

The role of the new symbols x and y is to mark the recursive part of the dynamics. Since $S \subseteq \Sigma^*$, no word of the seed contains either marker, whereas every word produced through the recursive term $(xLy)^q$ necessarily contains both x and y . Hence $S \cap (xLy)^q = \emptyset$ for every $L \subseteq \Gamma^*$. Thus, once information from a previous Picard iterate passes through the guard, it remains distinguishable from the newly inserted seed contribution.

The results below show that this separation has a strong consequence: the initial language remains exactly traceable and recoverable from every finite Picard iterate. This will ultimately allow us to classify the finite orbit according to the exact position of the initial language in the Chomsky hierarchy, while the limiting fixed point remains context-free.

Remark 4.2. Since the marked operator is a special case of the general guarded q -power construction over the ambient alphabet Γ , all fixed-point results from the preceding section remain applicable. In particular, whenever $S \in \mathcal{L}_{\text{cf}}$, the operator has a unique context-free fixed point $L_q^{(*)}$, and every Picard orbit converges to this same fixed point independently of the initial language.

For the remainder of this section, we use the following conventions. Let T_{qm} be the marked operator from Definition 4.1, with fixed integer $q \geq 1$ and seed $S \subseteq \Sigma^*$. The alphabets Σ and $\Gamma = \Sigma \cup \{x, y\}$ are as specified in that definition. We write $\mathcal{L} = \mathcal{P}(\Gamma^*)$, and all language classes are understood over Γ . For an arbitrary initial language $H \subseteq \Sigma^*$, its Picard orbit is denoted by $L^{(0)} = H$, $L^{(n+1)} = T_{qm}(L^{(n)})$, $n \geq 0$. The unique fixed point of T_{qm} is denoted by $L_q^{(*)}$.

For each word $z \in \Sigma^*$, define the auxiliary words z_n, α_n, β_n by $z_0 = z$, $\alpha_0 = \beta_0 = \varepsilon$, and $z_{n+1} = (xz_n y)^q$, $\alpha_{n+1} = x\alpha_n$, $\beta_{n+1} = \beta_n y (xz_n y)^{q-1}$, $n \geq 0$. Here $w^0 = \varepsilon$ for every word w . We also set $R_n = \alpha_n \Sigma^* \beta_n$ for every $n \geq 0$.

Theorem 4.3. *Let $H \neq \emptyset$ and choose $z \in H$. With the notation above, for every $n \geq 0$, one has $z_n \in L^{(n)}$, the language R_n is regular, and*

$$L^{(n)} \cap R_n = \alpha_n H \beta_n. \quad (4.1)$$

Proof. We first show that $z_n \in L^{(n)}$ for every $n \geq 0$. Since $z_0 = z \in H = L^{(0)}$, the assertion holds for $n = 0$. If $z_n \in L^{(n)}$, then $z_{n+1} = (xz_n y)^q \in (xL^{(n)}y)^q \subseteq L^{(n+1)}$. Hence the assertion follows by induction.

Since $R_n = \alpha_n \Sigma^* \beta_n$ is obtained from the regular language Σ^* by concatenation with the two fixed words α_n and β_n , the language R_n is regular for every $n \geq 0$.

Let $\pi : \Gamma^* \rightarrow \{x, y\}^*$ be the homomorphism that erases every symbol of the original alphabet Σ and keeps only the symbols x and y . For a word $v \in \{x, y\}^*$, define its balance by $\delta(v) = |v|_x - |v|_y$. We establish two structural properties.

First, for every $k \geq 0$ and every $w \in L^{(k)}$, the projection $\pi(w)$ is a Dyck word: its total balance is zero and every prefix has nonnegative balance.

For $k = 0$, one has $L^{(0)} = H \subseteq \Sigma^*$. Since neither x nor y occurs in any word of H , one has $\pi(w) = \varepsilon$ for every $w \in L^{(0)}$.

Assume that the assertion holds for every word of $L^{(k)}$. Since $S \subseteq \Sigma^*$, every seed word $s \in S$ also satisfies $\pi(s) = \varepsilon$. Every other word of $L^{(k+1)}$ belongs to $(xL^{(k)}y)^q$ and is therefore a concatenation of q blocks of the form xwy , with $w \in L^{(k)}$. By the induction hypothesis, $\pi(w)$ is a Dyck word, and therefore $x\pi(w)y$ is again a Dyck word. A concatenation of q such words is also a Dyck word. Hence the assertion holds for $L^{(k+1)}$.

Moreover, if $w \in L^{(k)}$, then $x\pi(w)y$ has strictly positive balance on every proper nonempty prefix and returns to balance zero only at its final symbol. Consequently, every word in $(xL^{(k)}y)^q$ has a unique decomposition into its q top-level factors $xw_1y, \dots, xw_qy$, $w_i \in L^{(k)}$, and the factor boundaries are determined uniquely by the successive returns of the x - y balance to zero.

Second, we claim that $\pi(r) = \pi(z_n)$ for every $r \in R_n$.

For $n = 0$, one has $R_0 = \Sigma^*$, while $z_0 = z \in H \subseteq \Sigma^*$. Hence $\pi(r) = \varepsilon = \pi(z_0)$ for every $r \in R_0$.

Assume that the claim holds for some $n \geq 0$. From the recursive definitions of α_n and β_n , $R_{n+1} = xR_ny(xz_ny)^{q-1}$. Thus every $r' \in R_{n+1}$ has the form $r' = xry(xz_ny)^{q-1}$ for some $r \in R_n$. Using the induction hypothesis gives

$$\pi(r') = x\pi(r)y(x\pi(z_n)y)^{q-1} = (x\pi(z_n)y)^q = \pi(z_{n+1}).$$

This proves the second structural property.

We now prove (4.1) by induction on n .

For $n = 0$, one has $R_0 = \Sigma^*$ and $L^{(0)} = H \subseteq \Sigma^*$. Therefore $L^{(0)} \cap R_0 = H = \alpha_0 H \beta_0$.

Assume that (4.1) holds for some $n \geq 0$. Since $R_{n+1} = xR_ny(xz_ny)^{q-1}$, every word of R_{n+1} contains the new symbol x , whereas every word of the seed S belongs to Σ^* and therefore contains neither x nor y . Consequently, $R_{n+1} \cap S = \emptyset$. Hence every word in $L^{(n+1)} \cap R_{n+1}$ must belong to the recursive part $(xL^{(n)}y)^q$.

Every word in R_{n+1} has the form $xry(xz_ny)^{q-1}$, $r \in R_n$. By the second structural property, $\pi(r) = \pi(z_n)$. Since $\pi(z_n)$ is a Dyck word, the projection of xry is primitive: its balance is strictly positive on every proper nonempty prefix and returns to zero only at the final y . Therefore xry occupies exactly one top-level block in the unique decomposition described above. When $q > 1$, it is followed by exactly $q - 1$ top-level blocks equal to xz_ny .

Suppose that the displayed word also belongs to $(xL^{(n)}y)^q$. By uniqueness of the top-level decomposition, its first interior word must satisfy $r \in L^{(n)}$,

while the remaining $q - 1$ interior words are $z_n \in L^{(n)}$. Since also $r \in R_n$, we obtain $L^{(n+1)} \cap R_{n+1} \subseteq x(L^{(n)} \cap R_n)y(xz_ny)^{q-1}$.

Conversely, let $r \in L^{(n)} \cap R_n$. Since $z_n \in L^{(n)}$, $xry(xz_ny)^{q-1} \in (xL^{(n)}y)^q$. The same word belongs to $R_{n+1} = xR_ny(xz_ny)^{q-1}$, and hence $xry(xz_ny)^{q-1} \in L^{(n+1)} \cap R_{n+1}$. Therefore $L^{(n+1)} \cap R_{n+1} = x(L^{(n)} \cap R_n)y(xz_ny)^{q-1}$.

Applying the induction hypothesis gives

$$L^{(n+1)} \cap R_{n+1} = x\alpha_n H \beta_n y(xz_ny)^{q-1} = \alpha_{n+1} H \beta_{n+1}.$$

Thus (4.1) holds for every $n \geq 0$. $\square$

The preceding trace identity allows us to go one step further: the initial language can be reconstructed exactly from every finite Picard iterate. Thus no information about the initial state is lost at any finite stage of the dynamics.

To compare initial languages containing a common distinguished word, for each $z \in \Sigma^*$ we set $\mathfrak{H}_z = \{H \subseteq \Sigma^* : z \in H\}$. For $H \in \mathfrak{H}_z$, we write $L_H^{(0)} = H$ and $L_H^{(n+1)} = T_{qm}(L_H^{(n)})$ to indicate explicitly the dependence of the Picard orbit on H . For every $n \geq 0$, define $F_n : \mathfrak{H}_z \rightarrow \mathcal{P}(\Gamma^*)$ by $F_n(H) = L_H^{(n)}$. With z fixed, the auxiliary words z_n, α_n, β_n and the regular language R_n are the same for all $H \in \mathfrak{H}_z$.

Theorem 4.4. *Let $z \in \Sigma^*$. For every $H \in \mathfrak{H}_z$ and $n \geq 0$, the initial language is recovered from its Picard iterate by*

$$H = \left(\alpha_n^{-1} (L^{(n)} \cap R_n) \right) \beta_n^{-1}. \quad (4.2)$$

Consequently, the map $F_n : \mathfrak{H}_z \rightarrow \mathcal{P}(\Gamma^*)$ is injective for every $n \geq 0$.

Proof. Let $H \subseteq \Sigma^*$ be nonempty and choose $z \in H$. By Theorem 4.3,

$$L^{(n)} \cap R_n = \alpha_n H \beta_n \quad \text{for every } n \geq 0.$$

Taking the left quotient by the fixed word α_n gives

$$\alpha_n^{-1} (L^{(n)} \cap R_n) = \alpha_n^{-1} (\alpha_n H \beta_n) = H \beta_n.$$

Indeed, cancellation in the free monoid implies that $\alpha_n w \in \alpha_n H \beta_n$ if and only if $w \in H \beta_n$. Taking next the right quotient by the fixed word β_n yields

$$\left(\alpha_n^{-1} (L^{(n)} \cap R_n) \right) \beta_n^{-1} = (H \beta_n) \beta_n^{-1} = H.$$

This proves (4.2).

We now prove the injectivity statement. Fix $z \in \Sigma^*$ and let $H_1, H_2 \in \mathfrak{H}_z$. Since the same distinguished word z belongs to both languages, the sequences z_n, α_n, β_n and the regular language $R_n = \alpha_n \Sigma^* \beta_n$ are identical for the two Picard orbits.

Suppose that, for some finite n , $F_n(H_1) = F_n(H_2)$. Applying the recovery formula (4.2) to this common finite iterate gives

$$H_1 = (\alpha_n^{-1}(F_n(H_1) \cap R_n)) \beta_n^{-1} = (\alpha_n^{-1}(F_n(H_2) \cap R_n)) \beta_n^{-1} = H_2.$$

Therefore F_n is injective for every finite n . $\square$

Corollary 4.5. *Under the assumptions of Theorem 4.4, every finite-time map*

$$F_n : \mathfrak{H}_z \rightarrow \mathcal{P}(\Gamma^*)$$

is injective, while

$$\sup_{H \in \mathfrak{H}_z} d(F_n(H), L_q^{(*)}) \leq 2^{-2qn}. \quad (4.3)$$

Consequently, the sequence of injective maps F_n converges uniformly on $\mathfrak{H}_z$ to the constant map $F_\infty(H) = L_q^{(*)}$.

Proof. Injectivity follows from Theorem 4.4. In the present construction the single guard is (x, y) , so the minimum guard length is $m = 2$. The contraction estimate from Theorem 2.10 therefore gives $d(F_n(H), L_q^{(*)}) \leq 2^{-2qn} d(H, L_q^{(*)}) \leq 2^{-2qn}$, because the language ultrametric takes values in $[0, 1]$. The bound is independent of H , which proves (4.3) and hence uniform convergence to the constant map $H \mapsto L_q^{(*)}$. $\square$

Remark 4.6. Corollary 4.5 separates finite-time information from asymptotic behaviour. At every finite stage the Picard dynamics still distinguishes all initial languages in $\mathfrak{H}_z$, since F_n is injective and the initial language can be reconstructed exactly. Nevertheless, the same maps converge uniformly to a constant map. Thus no finite stage loses the encoded initial information, although this information disappears completely from the limit state.

We now turn to the preservation of language classes under the marked Picard iteration. For a class $\mathcal{C} \subseteq \mathcal{L}$, we consider the following recovery closure properties:

- (R1) If $L \in \mathcal{C}$ and $R \subseteq \Gamma^*$ is regular, then $L \cap R \in \mathcal{C}$.
- (R2) If $L \in \mathcal{C}$ and $u, v \in \Gamma^*$ are fixed words, then $u^{-1}L \in \mathcal{C}$ and $Lv^{-1} \in \mathcal{C}$.

Theorem 4.7. *Let $H \neq \emptyset$, and let $\mathcal{C} \subseteq \mathcal{L}$ satisfy (R1)–(R2).*

Then, for every $n \geq 0$,

$$L^{(n)} \in \mathcal{C} \implies H \in \mathcal{C} \quad (4.4)$$

and

$$H \notin \mathcal{C} \implies L^{(n)} \notin \mathcal{C}. \quad (4.5)$$

If, in addition,

$$T_{qm}(\mathcal{C}) \subseteq \mathcal{C}, \quad (4.6)$$

then, for every $n \geq 0$,

$$H \in \mathcal{C} \iff L^{(n)} \in \mathcal{C}. \quad (4.7)$$

Proof. We first prove the backward implication (4.4). Let $n \geq 0$ be fixed and suppose that $L^{(n)} \in \mathcal{C}$.

By Theorem 4.4, there exist fixed words $\alpha_n, \beta_n \in \Gamma^*$ and a regular language $R_n = \alpha_n \Sigma^* \beta_n$ such that

$$H = \left(\alpha_n^{-1} (L^{(n)} \cap R_n) \right) \beta_n^{-1}. \quad (4.8)$$

Since R_n is regular and $L^{(n)} \in \mathcal{C}$, property (R1) gives $L^{(n)} \cap R_n \in \mathcal{C}$. Applying the left quotient by the fixed word α_n and using property (R2), we obtain $\alpha_n^{-1} (L^{(n)} \cap R_n) \in \mathcal{C}$. A second application of (R2), now to the right quotient by the fixed word β_n , gives $(\alpha_n^{-1} (L^{(n)} \cap R_n)) \beta_n^{-1} \in \mathcal{C}$. By (4.8), the language on the left-hand side is exactly H . Hence $H \in \mathcal{C}$. This proves (4.4).

Taking the contrapositive immediately yields $H \notin \mathcal{C} \implies L^{(n)} \notin \mathcal{C}$ for every finite $n \geq 0$, which proves (4.5).

Assume now, in addition, that $T_{qm}(\mathcal{C}) \subseteq \mathcal{C}$. Suppose first that $H \in \mathcal{C}$. Since $L^{(0)} = H$, we have $L^{(0)} \in \mathcal{C}$. If $L^{(k)} \in \mathcal{C}$ for some $k \geq 0$, then the invariance assumption gives $L^{(k+1)} = T_{qm}(L^{(k)}) \in \mathcal{C}$. Therefore, by induction, $H \in \mathcal{C} \implies L^{(n)} \in \mathcal{C}$ for every $n \geq 0$.

The converse implication has already been proved in (4.4). Hence, for every finite $n \geq 0$, $H \in \mathcal{C} \iff L^{(n)} \in \mathcal{C}$. This proves (4.7). $\square$

Remark 4.8. Theorem 4.7 separates two independent mechanisms. The recovery closure properties prevent a finite Picard iterate from entering a class that does not contain the initial language, while forward invariance prevents an orbit starting inside the class from leaving it. When both mechanisms are available, class membership becomes an exact finite-time invariant of the dynamics.

The abstract preservation principle established above can now be applied to the successive classes of the Chomsky hierarchy. This allows us to determine not merely whether a finite Picard iterate belongs to a given language class, but whether it remains in the same exact layer of the hierarchy as the initial language.

For the classification below, we use the Chomsky classes from Definition 2.13 for languages over Γ and set

$$\begin{aligned}\text{Type}(-1) &= \mathcal{L}, & \text{Type}(0) &= \mathcal{L}_{\text{un}}, \\ \text{Type}(1) &= \mathcal{L}_{\text{cs}}, & \text{Type}(2) &= \mathcal{L}_{\text{cf}}, \\ \text{Type}(3) &= \mathcal{L}_{\text{reg}}, & \text{Type}(4) &= .\end{aligned}$$

The exact layers of the extended hierarchy are $\mathcal{D}_i = \text{Type}(i) \setminus \text{Type}(i+1)$, where $i \in \{-1, 0, 1, 2, 3\}$.

Theorem 4.9. *For the marked Picard orbit of T_{qm} the following assertions hold.*

(i) *If $S \in \mathcal{L}_{\text{cf}}$, then for every $i \in \{-1, 0, 1, 2\}$ and $n \geq 0$,*

$$L^{(0)} \in \mathcal{D}_i \implies L^{(n)} \in \mathcal{D}_i. \quad (4.9)$$

(ii) *If $S \in \mathcal{L}_{\text{reg}}$, then for every $i \in \{-1, 0, 1, 2, 3\}$ and $n \geq 0$,*

$$L^{(0)} \in \mathcal{D}_i \implies L^{(n)} \in \mathcal{D}_i. \quad (4.10)$$

In both cases, $L^{(n)} \rightarrow L_q^{(*)} \in \mathcal{L}_{\text{cf}}$, and the limit depends only on T_{qm} , not on the initial language $L^{(0)}$.

Proof. We apply Theorem 4.7 to the successive classes of the extended Chomsky hierarchy.

By Theorem 2.14, each of the classes $\mathcal{L}_{\text{reg}}$, $\mathcal{L}_{\text{cf}}$, $\mathcal{L}_{\text{cs}}$, $\mathcal{L}_{\text{un}}$ satisfies the recovery assumptions (R1)–(R2) of Theorem 4.7. The same is trivially true for $\text{Type}(-1) = \mathcal{P}(\Gamma^*)$.

Furthermore, Theorem 2.14 implies that whenever $S \in \text{Type}(i)$, the class $\text{Type}(i)$ is invariant under T_{qm} . Indeed, if $L \in \text{Type}(i)$, then $(xLy)^q \in \text{Type}(i)$, and hence $T_{qm}(L) = S \cup (xLy)^q \in \text{Type}(i)$.

We first prove part (i). Assume that $S \in \mathcal{L}_{\text{cf}} = \text{Type}(2)$.

Consider first $i = -1$. If $H = L^{(0)} \in \mathcal{D}_{-1}$, then $H \notin \mathcal{L}_{\text{un}}$. Since $\mathcal{L}_{\text{un}}$ satisfies the recovery assumptions of Theorem 4.7, its contrapositive part gives $L^{(n)} \notin \mathcal{L}_{\text{un}}$ for every finite $n \geq 0$. Since every $L^{(n)} \in \text{Type}(-1) = \mathcal{L}$, we obtain $L^{(n)} \in \mathcal{D}_{-1}$ for every finite $n \geq 0$.

Now let $i \in \{0, 1, 2\}$ and suppose that $H = L^{(0)} \in \mathcal{D}_i$. Then $H \in \text{Type}(i)$ and $H \notin \text{Type}(i+1)$. Since $S \in \text{Type}(2)$, the inclusions of the Chomsky hierarchy imply $S \in \text{Type}(i)$. Hence $\text{Type}(i)$ is invariant under T_{qm} , and Theorem 4.7 gives $L^{(n)} \in \text{Type}(i)$ for every finite $n \geq 0$.

On the other hand, $\text{Type}(i+1)$ satisfies the recovery assumptions of the same theorem. Since $H \notin \text{Type}(i+1)$, the contrapositive implication gives $L^{(n)} \notin$

$\text{Type}(i+1)$ for every finite $n \geq 0$. Therefore $L^{(n)} \in \text{Type}(i) \setminus \text{Type}(i+1) = \mathcal{D}_i$ for every finite $n \geq 0$. This proves part (i).

We next prove part (ii). Assume that $S \in \mathcal{L}_{\text{reg}} = \text{Type}(3)$. Then $S \in \text{Type}(i)$ for every $i \in \{-1, 0, 1, 2, 3\}$.

For $i \in \{-1, 0, 1, 2\}$, the argument from part (i) applies without change and gives $H \in \mathcal{D}_i \implies L^{(n)} \in \mathcal{D}_i$ for every finite $n \geq 0$.

It remains to consider $i = 3$. Since $\mathcal{D}_3 = \text{Type}(3) \setminus \text{Type}(4) = \mathcal{L}_{\text{reg}}$, let $H = L^{(0)} \in \mathcal{L}_{\text{reg}}$. Because S is regular and the regular languages are closed under concatenation and finite union, the class $\mathcal{L}_{\text{reg}}$ is invariant under T_{qm} . Therefore $L^{(n)} \in \mathcal{L}_{\text{reg}} = \mathcal{D}_3$ for every $n \geq 0$. This argument also covers the case $H =$.

Finally, in part (i) the seed is context-free by assumption, while in part (ii) it is regular and hence context-free. Consequently, Theorem 3.1 gives $L_q^{(*)} \in \mathcal{L}_{\text{cf}}$. By the contraction theorem, $L^{(n)} \longrightarrow L_q^{(*)}$ for every initial language $L^{(0)}$. Since the fixed point is unique, it depends only on the operator T_{qm} and not on the initial language.

Thus every exact Chomsky layer specified in parts (i) and (ii) is preserved at every finite Picard stage, whereas all corresponding orbits converge to the same context-free fixed point. $\square$

Corollary 4.10. *Assume that $S \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$. For the marked operator $T_{qm}(L) = S \cup (xLy)^q$, the unique fixed point satisfies $L_q^{(*)} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$.*

Proof. By Theorem 3.1, $L_q^{(*)} \in \mathcal{L}_{\text{cf}}$. Since $L_q^{(*)} = S \cup (xL_q^{(*)}y)^q$, every word in the recursive part contains the new symbols x and y , whereas every word of S belongs to Σ^* . Consequently, $L_q^{(*)} \cap \Sigma^* = S$. If $L_q^{(*)}$ were regular, then, since Σ^* is regular and the regular languages are closed under intersection, the language $S = L_q^{(*)} \cap \Sigma^*$ would also be regular. This contradicts $S \notin \mathcal{L}_{\text{reg}}$. Hence $L_q^{(*)} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$. $\square$

Corollary 4.11. *Assume that $S \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$ and that $L^{(0)} = H \in \mathcal{L}_{\text{reg}}$. For the marked operator $T_{qm}(L) = S \cup (xLy)^q$, one has $L^{(0)} \in \mathcal{L}_{\text{reg}}$, while $L^{(n)} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$ for every $n \geq 1$.*

Proof. Since both S and H are context-free, closure under concatenation and finite union gives $L^{(n)} \in \mathcal{L}_{\text{cf}}$ for every $n \geq 0$.

For every $n \geq 1$, the recursive part of $L^{(n)} = S \cup (xL^{(n-1)}y)^q$ contains the fresh marker symbols x and y , whereas every word of S belongs to Σ^* . Consequently, $L^{(n)} \cap \Sigma^* = S$ for every $n \geq 1$.

Suppose that $L^{(n)}$ were regular for some $n \geq 1$. Since Σ^* is regular and the regular languages are closed under intersection, it would follow that $S = L^{(n)} \cap \Sigma^*$ is regular, contradicting $S \notin \mathcal{L}_{\text{reg}}$. Hence every $L^{(n)}$, $n \geq 1$, is context-free but nonregular. $\square$

The conclusions of Theorem 4.9 and Corollaries 4.10 and 4.11 are summarized in Tables 4.1 and 4.2. In each table, the first column gives the class of the initial language, the second describes the finite Picard iterates, and the third gives the class of the common fixed point. The first table covers the case of a regular seed, while the second covers the case of a genuinely context-free seed.

TABLE 4.1. Case I: Classification of the finite Picard iterates and of the fixed point when $S \in \mathcal{L}_{\text{reg}}$.

Class of $L^{(0)}$	Class of $L^{(n)}$	Class of $L_q^{(*)}$
$\mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{cf}}$
$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{cf}}$
$\mathcal{L}_{\text{cs}} \setminus \mathcal{L}_{\text{cf}}$	$\mathcal{L}_{\text{cs}} \setminus \mathcal{L}_{\text{cf}}$	$\mathcal{L}_{\text{cf}}$
$\mathcal{L}_{\text{un}} \setminus \mathcal{L}_{\text{cs}}$	$\mathcal{L}_{\text{un}} \setminus \mathcal{L}_{\text{cs}}$	$\mathcal{L}_{\text{cf}}$
$\mathcal{L}_{\text{nrn}} = \mathcal{L} \setminus \mathcal{L}_{\text{un}}$	$\mathcal{L}_{\text{nrn}}$	$\mathcal{L}_{\text{cf}}$

TABLE 4.2. Case II: Classification of the finite Picard iterates and of the fixed point when $S \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$.

Class of $L^{(0)}$	Class of $L^{(n)}$ $n \geq 1$	Class of $L_q^{(*)}$
$\mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$
$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$
$\mathcal{L}_{\text{cs}} \setminus \mathcal{L}_{\text{cf}}$	$\mathcal{L}_{\text{cs}} \setminus \mathcal{L}_{\text{cf}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$
$\mathcal{L}_{\text{un}} \setminus \mathcal{L}_{\text{cs}}$	$\mathcal{L}_{\text{un}} \setminus \mathcal{L}_{\text{cs}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$
$\mathcal{L}_{\text{nrn}} = \mathcal{L} \setminus \mathcal{L}_{\text{un}}$	$\mathcal{L}_{\text{nrn}}$	$\mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$

5. EXAMPLES

In each of the following examples, we take $q = 2$ and denote the corresponding marked operator T_{qm} from Definition 4.1 by T_{2m} .

We conclude with four examples illustrating the main structural phenomena established above. The first gives an explicit context-free fixed point for the marked q -power dynamics. The second illustrates the finite-time trace and exact recovery of the initial language. The third shows preservation of every exact layer of the extended Chomsky hierarchy when the seed is regular. The fourth considers a genuinely context-free seed and a regular initial language, showing that the orbit leaves the regular class after the first Picard step and remains genuinely context-free at every subsequent finite stage.

Example 5.1. Let $\Sigma = \{c\}$, let $x, y \notin \Sigma$ be two distinct new symbols, and put $\Gamma = \Sigma \cup \{x, y\}$. Take the regular seed $S = \{c\}$, let $q = 2$, and consider the operator $T_{2m} : \mathcal{P}(\Gamma^*) \rightarrow \mathcal{P}(\Gamma^*)$ defined by $T_{2m}(L) = \{c\} \cup (xLy)^2$. The single guard is (x, y) , so its length is $m = |x| + |y| = 2$. Hence Theorem 2.10 gives $d(T_{2m}(L), T_{2m}(M)) \leq 2^{-4}d(L, M)$, and therefore T_{2m} has a unique fixed point $L_2^{(*)}$.

Since the seed is finite and hence context-free, Theorem 3.1 implies that $L_2^{(*)} \in \mathcal{L}_{cf}$. The fixed-point grammar can be written particularly simply. Introduce two nonterminals X and B , take X as the start symbol, and use the productions

$$X \rightarrow c \mid BB, \quad B \rightarrow xXy.$$

If $K = L(X)$, then B generates xKy , while $X \rightarrow BB$ generates $(xKy)^2$. Consequently, $K = \{c\} \cup (xKy)^2 = T_{2m}(K)$. By uniqueness of the fixed point, $K = L_2^{(*)}$.

For instance, starting from $L^{(0)} = \{c\}$ gives $L^{(1)} = \{c\}$ and $L^{(2)} = \{c, xcyxycy\}$. The finite iterates depend on the chosen initial language, whereas the fixed point does not. Every Picard orbit of this operator converges to the same context-free language $L_2^{(*)}$.

This example illustrates the first main phenomenon of the paper: the language class of the limiting fixed point is determined by the operator and its seed, independently of the language from which the Picard iteration starts.

Example 5.2. We next illustrate Theorems 4.3 and 4.4. Let $\Sigma = \{a, s\}$, let $x, y \notin \Sigma$ be two distinct new symbols, and put $\Gamma = \Sigma \cup \{x, y\}$. Take the regular seed $S = \{s\}$, let $q = 2$, and define $T_{2m}(L) = \{s\} \cup (xLy)^2$. Let $H = L^{(0)} = \{a, aa\} \subseteq \Sigma^*$ and choose the distinguished word $z = a \in H$.

At time $n = 0$, one has $z_0 = a$, $\alpha_0 = \beta_0 = \varepsilon$, and $R_0 = \Sigma^*$. Hence $L^{(0)} \cap R_0 = H$.

At the first Picard step, $xHy = \{xay, xaay\}$, and therefore

$$L^{(1)} = \{s, xayxay, xayxaay, xaayxay, xaayxaay\}.$$

The distinguished word evolves according to $z_1 = (xay)^2 = xayxay$. Moreover, $\alpha_1 = x$ and $\beta_1 = y(xay) = yxay$. Thus $R_1 = x\Sigma^*yxay$.

Among the words of $L^{(1)}$, exactly $xayxay$ and $xaayxay$ belong to R_1 . Hence

$$L^{(1)} \cap R_1 = \{xayxay, xaayxay\} = x\{a, aa\}yxay = \alpha_1 H \beta_1.$$

The initial language is therefore recovered exactly by

$$x^{-1}(L^{(1)} \cap R_1)(yxay)^{-1} = \{a, aa\} = H.$$

At the second step, $\alpha_2 = xx$ and $\beta_2 = \beta_1 y(xz_1 y)$. Hence $R_2 = xx\Sigma^*\beta_2$. The trace identity gives directly $L^{(2)} \cap R_2 = xxH\beta_2 = \{xxa\beta_2, xxaa\beta_2\}$. Applying the left quotient by xx and the right quotient by β_2 again recovers H .

The same mechanism holds at every finite stage: $L^{(n)} \cap R_n = \alpha_n H \beta_n$, and consequently $H = (\alpha_n^{-1}(L^{(n)} \cap R_n))\beta_n^{-1}$. Thus the initial language remains exactly encoded in every finite Picard iterate, even though the orbit converges to a fixed point that is completely independent of H .

The example also makes the injectivity result transparent. If $H_1, H_2 \subseteq \Sigma^*$ contain the same distinguished word z and satisfy $T_{2m}^n(H_1) = T_{2m}^n(H_2)$ for some finite n , then the same regular slice and the same fixed-word quotients recover both initial languages. Hence $H_1 = H_2$.

Example 5.3. We now illustrate Theorem 4.9 simultaneously across the extended Chomsky hierarchy.

Let $\Sigma = \{a, b, c, s\}$, let $x, y \notin \Sigma$ be distinct new symbols, and put $\Gamma = \Sigma \cup \{x, y\}$. Take the regular seed $S = \{s\}$, let $q = 2$, and consider $T_{2m}(L) = \{s\} \cup (xLy)^2$. Since the seed is regular, part (ii) of Theorem 4.9 applies to every layer of the extended hierarchy.

Consider first the regular initial language $H_3 = a^*$. If $L_{H_3}^{(0)} = H_3$, then $L_{H_3}^{(n)} \in \mathcal{L}_{\text{reg}}$ for every finite $n \geq 0$.

Next, let $H_2 = \{a^k b^k : k \geq 1\}$. This language is context-free but not regular. Therefore $L_{H_2}^{(n)} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$ for every finite $n \geq 0$, where $L_{H_2}^{(0)} = H_2$.

Now take $H_1 = \{a^k b^k c^k : k \geq 1\}$. The language H_1 is context-sensitive but not context-free. Hence, if $L_{H_1}^{(0)} = H_1$, then $L_{H_1}^{(n)} \in \mathcal{L}_{\text{cs}} \setminus \mathcal{L}_{\text{cf}}$ for every finite $n \geq 0$.

By the strictness of the Chomsky hierarchy, choose a nonempty language $H_0 \in \mathcal{L}_{\text{un}} \setminus \mathcal{L}_{\text{cs}}$. Then the corresponding orbit satisfies $L_{H_0}^{(n)} \in \mathcal{L}_{\text{un}} \setminus \mathcal{L}_{\text{cs}}$ for every finite $n \geq 0$.

Finally, choose a language $H_{-1} \in \mathcal{P}(\Sigma^*) \setminus \mathcal{L}_{\text{un}}$. For the Picard orbit starting from H_{-1} , Theorem 4.9 yields $L_{H_{-1}}^{(n)} \in \mathcal{P}(\Gamma^*) \setminus \mathcal{L}_{\text{un}}$ for every finite $n \geq 0$.

Thus, with a regular seed, every exact layer of the extended Chomsky hierarchy is preserved at every finite Picard step. In particular, the finite dynamics cannot move an initial language into a strictly smaller Chomsky class.

The limiting behaviour is fundamentally different. Since the seed $S = \{s\}$ is context-free, Theorem 3.1 gives $L_2^{(*)} \in \mathcal{L}_{\text{cf}}$. All five Picard orbits above converge to this same fixed point. Consequently, for example, $L_{H_1}^{(n)} \in \mathcal{L}_{\text{cs}} \setminus \mathcal{L}_{\text{cf}}$ for every finite n , $L_{H_1}^{(n)} \rightarrow L_2^{(*)} \in \mathcal{L}_{\text{cf}}$. Likewise, an orbit may remain outside the recursively enumerable class at every finite stage and nevertheless converge to the same context-free fixed point.

This is the finite-time versus limiting phenomenon captured by the central theorem: the exact Chomsky layer of the initial language is preserved throughout every finite stage, while the limiting language may belong to a strictly smaller class.

Example 5.4. The preceding example uses a regular seed and shows preservation of every exact layer of the extended Chomsky hierarchy. We now illustrate the different behaviour produced by a genuinely context-free seed when the initial language is regular.

Let $\Sigma = \{a, b\}$, let $x, y \notin \Sigma$ be two distinct fresh symbols, and put $\Gamma = \Sigma \cup \{x, y\}$. Take $S = \{a^k b^k : k \geq 1\} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$, let $q = 2$, and consider $T_{2m}(L) = S \cup (xLy)^2$. Choose the regular initial language $L^{(0)} = \{\varepsilon\}$.

At the first Picard step, $L^{(1)} = S \cup \{xyxy\}$, and therefore $L^{(1)} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$.

In fact, the same conclusion holds at every later finite step. For each $n \geq 1$, $L^{(n)} = S \cup (xL^{(n-1)}y)^2$. Every word in the recursive part contains the marker symbols x and y , whereas every word of S belongs to Σ^* . Hence $L^{(n)} \cap \Sigma^* = S$ for every $n \geq 1$. If some $L^{(n)}$, $n \geq 1$, were regular, then its intersection with the regular language Σ^* would also be regular. This would imply that S is regular, a contradiction. Consequently, $L^{(n)} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$ for every $n \geq 1$.

Thus a regular initial language leaves the regular class immediately when the seed is genuinely context-free, and all subsequent finite Picard iterates remain genuinely context-free. Moreover, Corollary 4.10 gives $L_2^{(*)} \in \mathcal{L}_{\text{cf}} \setminus \mathcal{L}_{\text{reg}}$, and the orbit converges to this same genuinely context-free fixed point.

This example illustrates Corollaries 4.10 and 4.11 and completes the classification summarized in Tables 4.1 and 4.2.

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БЕЗКОНТЕКСТНИ НЕПОДВИЖНИ ТОЧКИ И ПЪЛНА КЛАСИФИКАЦИЯ НА ОРБИТЕ ПРИ ИТЕРАЦИИТЕ НА PICARD ЗА СТЕПЕННИ ОПЕРАТОРИ С ГАРДОВЕ ВЪРХУ ФОРМАЛНИ ЕЗИЦИ

Атанас Илчев, Христо Кискинов, Георги Пашев, Боян Златанов

Резюме. Изследваме от гледна точка на теорията на формалните езици структурата на неподвижните точки и на езиците, получени след краен брой итерации на Picard, за q -степенни оператори с гардове. За общия оператор доказваме, че безконтекстен базов език определя единствена безконтекстна неподвижна точка, и даваме ефективно построение на безконтекстна граматика, която поражда тази неподвижна точка, независимо от началния език. След това разглеждаме маркиран частен случай с по един гард, при който два нови символа разграничават рекурсивния принос от приноса на базовия език. В този случай началният език може да бъде проследен и възстановен точно от всеки език, получен след краен брой итерации на Picard чрез сечение с регулярен език и последователно вземане на ляво и дясно частно по фиксирани думи. Оттук следват инективността на съответните итерационни изображения и абстрактен принцип за запазване на принадлежността към класове от езици при краен брой итерации. Като следствие получаваме класификация на езиците, получени след краен брой итерации на Пикар, според точното място на началния език в йерархията на Chomsky. Така точната сложност в смисъла на теорията на формалните езици се запазва на всеки краен етап, докато всички орбити на Picard са сходящи към една и съща безконтекстна неподвижна точка.