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REICH AND BIANCHINI CONTRACTIVITY FOR FINITE-OBSERVATION CONDITIONAL GUARDED LANGUAGE OPERATORS

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Abstract. We study finite-observation conditional guarded language operators on the space of all formal languages over a finite alphabet equipped with the length-based first-disagreement ultrametric. We establish a general directed depth criterion ensuring that such an operator satisfies the Reich inequality with arbitrary coefficients $a, b, c \geq 0$, $a + b + c < 1$. We then show that Reich and Bianchini contractivity are equivalent on the language space: every Reich mapping on an ultrametric space is Bianchini, while the dyadic structure of the language metric implies that every Bianchini mapping satisfies a Reich inequality with coefficients $a = b = c = 1/4$. For conditional guarded operators, we derive a depth–residual criterion reaching the critical dyadic coefficient $1/2$, which is excluded by the classical Kannan condition. We also obtain an exact a posteriori residual identity for Bianchini contractions with fixed points on ultrametric spaces, which identifies the fixed-point error with the residual and yields exact Picard error–displacement identities and finite-depth certification in the language setting. The previously established Kannan criterion is recovered under the stricter coefficient threshold below $1/2$.

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1. Introduction

Fixed-point methods provide a useful framework for recursive constructions in formal language theory. On the space $\mathcal{L} = \mathcal{P}(\Sigma^*)$, equipped with the length-based first-disagreement ultrametric, guarded language operators are Banach contractions because fixed word contexts shift every possible disagreement to greater word length [1]. This yields unique fixed languages and convergence of the corresponding Picard iterations.

In this paper, we consider finite-observation conditional guarded operators whose active branch is selected according to the membership of finitely many prescribed test words. The guarded branch operators satisfy explicit branch-wise Lipschitz estimates, and the assumptions of our main theorem ensure that their coefficients are strictly smaller than one. Nevertheless, switching between different branches may prevent the global operator from satisfying a Banach inequality. To treat this behavior, we use the general Reich condition, which combines the distance between the input languages with their two residual displacements.

The finite-observation conditional framework was recently introduced in [2], where depth–residual conditions were used to obtain the stronger max-type estimate

$$d(T(A), T(B)) \leq \alpha \max\{d(A, T(A)), d(B, T(B))\}, \qquad \alpha < \frac{1}{2},$$

and hence global Kannan contractivity, including examples that are Kannan-contractive but not Banach-contractive. The max-residual estimate itself is of the classical form introduced by Bianchini [3], for which the admissible coefficient range is $0 \leq \lambda < 1$. Bianchini and Reich contractive conditions belong to the classical families of fixed-point contractions systematically compared by Rhoades [4] and, subsequently, by Collaço and Silva [5]. More recently, Bisht and Petrov [6] extended Bianchini fixed-point theory to semimetric spaces with triangle functions, including ultrametric specializations and Picard convergence estimates. In the present dyadic language metric, however, every strict max-residual contraction automatically admits the coefficient $1/2$. This leads to two natural questions: whether the finite-observation construction can reach the critical dyadic value $1/2$, which is the endpoint excluded by the classical Kannan coefficient range, and how Bianchini and Reich contractivity are related in the ultrametric setting. Both questions are answered below.

Our main result gives explicit branchwise and cross-branch depth conditions under which a finite-observation conditional guarded language operator is a Reich mapping with arbitrary coefficients $a, b, c \geq 0$ satisfying $a + b + c < 1$. We also establish a structural equivalence between Reich and Bianchini contractivity on the length-based language ultrametric. A separate depth–residual criterion then extends the previously established Kannan regime [2] to the critical dyadic value 1/2. The classical Kannan conclusion is recovered under the stricter threshold below 1/2. The max-residual structure further yields an exact residual identity for Bianchini contractions with fixed points in arbitrary ultrametric spaces. This identifies the fixed-point error exactly with the residual and, along Picard iteration, with the current displacement; in the language setting, these identities specialize to finite-depth certification results.

2. Preliminaries

Throughout the paper, Σ denotes a finite nonempty alphabet, Σ^* is the set of all finite words over Σ, including the empty word ε, and $\mathcal{L} = \mathcal{P}(\Sigma^*)$ is the set of all formal languages over Σ [7, 8]. For $L \in \mathcal{L}$, let $\mathbf{1}_L : \Sigma^* \to \{0,1\}$ denote its characteristic function. For $u, v \in \Sigma^*$, we write $uLv = \{uwv : w \in L\}$.

Definition 2.1 ([9, 10])**.** For $A, B \in \mathcal{L}$, define

$$d(A,B) = \begin{cases} 0, & A = B, \\ 2^{-\ell(A,B)}, & A \neq B, \end{cases} \qquad \ell(A,B) = \min\bigl\{|w| : \mathbf{1}_A(w) \neq \mathbf{1}_B(w)\bigr\}.$$

Thus, the distance between two distinct languages is determined by the length of their shortest distinguishing word.

Theorem 2.2 ([9, 10])**.** *The function d is an ultrametric on $\mathcal{L}$, and $(\mathcal{L}, d)$ is complete. Moreover, for every integer $N \geq 0$, $d(A,B) \leq 2^{-N}$ if and only if $\mathbf{1}_A(w) = \mathbf{1}_B(w)$ for every $w \in \Sigma^*$ with $|w| < N$.*

We next recall the metric estimates for fixed word contexts and finite unions.

Proposition 2.3 ([1])**.** *For all $A, B, S \in \mathcal{L}$ and $u, v \in \Sigma^*$, $d(uAv, uBv) \leq 2^{-(|u|+|v|)} d(A,B)$ and $d(A \cup S, B \cup S) \leq d(A,B)$. More generally, if $d(X_r, Y_r) \leq \delta$ for $r = 1, \dots, p$, then $d\left(\bigcup_{r=1}^{p} X_r, \bigcup_{r=1}^{p} Y_r\right) \leq \delta$.*

Let $G = \{(u_r, v_r) : r = 1, \dots, p\} \subseteq \Sigma^* \times \Sigma^*$ be a finite nonempty guard family, and define $m(G) = \min_{1 \leq r \leq p}\bigl(|u_r| + |v_r|\bigr)$, $\qquad F_G(L) = \bigcup_{r=1}^{p} u_r L v_r$.

Proposition 2.4 ([1])**.** *Let G be a finite guard family with $m(G) = m \geq 1$, and let $S \in \mathcal{L}$ be fixed. Then the guarded language operator $T_{S,G}(L) = S \cup F_G(L)$ satisfies $d(T_{S,G}(A), T_{S,G}(B)) \leq 2^{-m} d(A,B)$ for all $A, B \in \mathcal{L}$.*

We finally recall the contractive condition used in the main results.

Definition 2.5 ([11])**.** Let (X, ρ) be a metric space. A mapping $T : X \to X$ is called a Reich mapping if there exist constants $a, b, c \geq 0$ satisfying $a+b+c < 1$ such that $\rho(Tx, Ty) \leq a\rho(x,y) + b\rho(x,Tx) + c\rho(y,Ty)$ for all $x, y \in X$.

We shall also use the classical max-residual contractive condition introduced by Bianchini.

Definition 2.6 ([3])**.** Let (X, ρ) be a metric space. A mapping $T : X \to X$ is called a Bianchini contraction if there exists a constant $\lambda \in [0, 1)$ such that

$$\rho(Tx, Ty) \leq \lambda \max\{\rho(x, Tx), \rho(y, Ty)\}$$

for all $x, y \in X$.

Theorem 2.7 ([3])**.** *Let (X, ρ) be a complete metric space, and let $T : X \to X$ be a Bianchini contraction. Then T has a unique fixed point $x^* \in X$, and the Picard iteration $x_{n+1} = T(x_n)$ converges to x^* for every $x_0 \in X$.*

For comparison, we also recall the classical Kannan condition.

Definition 2.8 ([12])**.** Let (X, ρ) be a metric space. A mapping $T : X \to X$ is called a Kannan mapping if there exists a constant $\alpha \in [0, 1/2)$ such that

$$\rho(Tx, Ty) \leq \alpha\big(\rho(x, Tx) + \rho(y, Ty)\big)$$

for all $x, y \in X$.

Theorem 2.9 ([11])**.** *Let (X, ρ) be a complete metric space, and let $T : X \to X$ be a Reich mapping. Then T has a unique fixed point $x^* \in X$. Moreover, for every $x_0 \in X$, the Picard iteration $x_{n+1} = T(x_n), \qquad n \geq 0$, converges to x^*.*

3. Main Results

We introduce finite-observation conditional guarded language operators and establish sufficient conditions under which they satisfy the general Reich inequality with arbitrary coefficients $a, b, c \geq 0$, $a + b + c < 1$. We then study the structural relation between Reich and Bianchini contractivity in ultrametric spaces and show that the two classes coincide on the length-based language space. Finally, we derive a direct depth–residual Bianchini criterion that reaches the critical dyadic coefficient $1/2$, thereby extending the Kannan criterion previously obtained in [2]. The classical Kannan regime is recovered under the stricter coefficient threshold below $1/2$.

Let $W = \{w_1, \dots, w_s\} \subseteq \Sigma^*$ be a finite nonempty set of test words. For every $L \in \mathcal{L}$, define $\mathrm{obs}_W(L) = (\mathbf{1}_L(w_1), \dots, \mathbf{1}_L(w_s)) \in \{0, 1\}^s$. Let $I = \{1, \dots, q\}$, where $q \geq 2$, and let $\varphi : \{0, 1\}^s \to I$ be a surjective selector. We put $\beta(L) = \varphi(\mathrm{obs}_W(L))$.

For every $i \in I$, let $S_i \in \mathcal{L}$, let $p_i \in \mathbb{N}_0$, and let $G_i = \{(u_{i,r}, v_{i,r}) : r = 1, \dots, p_i\} \subseteq \Sigma^* \times \Sigma^*$ be a finite, possibly empty, guard family. If $p_i = 0$, set $F_i(L) = \varnothing$ and $\kappa_i = 0$. If $p_i \geq 1$, define $F_i(L) = \bigcup_{r=1}^{p_i} u_{i,r} L v_{i,r}$, $m_i = \min_{1 \leq r \leq p_i}(|u_{i,r}| + |v_{i,r}|)$, and $\kappa_i = 2^{-m_i}$. In both cases, put $T_i(L) = S_i \cup F_i(L)$.

Definition 3.1. The mapping $T : \mathcal{L} \to \mathcal{L}$ defined by $T(L) = T_{\beta(L)}(L)$ is called a finite-observation conditional guarded language operator. For every $i \in I$, the set $\mathcal{C}_i = \{L \in \mathcal{L} : \beta(L) = i\}$ is called the i-th branch class.

The classes $\mathcal{C}_1, \dots, \mathcal{C}_q$ form a partition of $\mathcal{L}$. They are nonempty because every vector in $\{0,1\}^s$ is the observation vector of some language and φ is surjective.

Proposition 3.2. *For every $i \in I$ and all $A, B \in \mathcal{L}$, one has $d(T_i(A), T_i(B)) \leq \kappa_i d(A, B)$.*

Proof. Fix $i \in I$. If $p_i = 0$, then $F_i(A) = F_i(B) = \varnothing$, and hence $T_i(A) = S_i = T_i(B)$. Therefore $d(T_i(A), T_i(B)) = 0 = \kappa_i d(A, B)$.

Suppose that $p_i \geq 1$. For every $r = 1, \dots, p_i$, Proposition 2.3 gives

$$d(u_{i,r}Av_{i,r}, u_{i,r}Bv_{i,r}) \leq 2^{-(|u_{i,r}|+|v_{i,r}|)} d(A, B) \leq 2^{-m_i} d(A, B).$$

Applying the finite-union estimate to the guarded terms, we obtain $d(F_i(A), F_i(B)) \leq 2^{-m_i} d(A, B)$. Since the same seed language is adjoined to both guarded cores,

$$d(T_i(A), T_i(B)) = d(S_i \cup F_i(A), S_i \cup F_i(B)) \leq d(F_i(A), F_i(B)) \leq \kappa_i d(A, B).$$

□

The following observation gives a uniform lower bound for the distance between languages belonging to different branch classes. For $i \neq j$, put $V_i = \varphi^{-1}(i)$ and $V_j = \varphi^{-1}(j)$. For $\eta \in V_i$ and $\xi \in V_j$, define $\pi(\eta, \xi) = \min\{|w_k| : \eta_k \neq \xi_k\}$, and set $\pi_{ij} = \max_{\eta \in V_i,\, \xi \in V_j} \pi(\eta, \xi)$.

Proposition 3.3. *For every $i \neq j$, $A \in \mathcal{C}_i$, and $B \in \mathcal{C}_j$, one has $d(A, B) \geq 2^{-\pi_{ij}}$.*

Proof. Let $\eta = \mathrm{obs}_W(A) \in V_i$ and $\xi = \mathrm{obs}_W(B) \in V_j$. Since $i \neq j$, the vectors η and ξ are distinct. Hence A and B disagree on an observation word of length $\pi(\eta, \xi) \leq \pi_{ij}$. Therefore their shortest distinguishing word has length at most π_{ij}, and consequently $d(A, B) \geq 2^{-\pi_{ij}}$. □

Consequently, whenever the input-distance term is used in a directed Reich pattern for the ordered pair $(\mathcal{C}_i, \mathcal{C}_j)$, one may take $\ell = \pi_{ij}$.

We now establish the abstract switching principle used in the main theorem.

Proposition 3.4. *Let (X, ρ) be an ultrametric space partitioned into nonempty sets $C_1, \dots, C_q$. For each $i \in I$, let $T_i : X \to X$, and define $T(x) = T_i(x)$ whenever $x \in C_i$. Suppose that there exist $a, b, c \geq 0$, $a + b + c < 1$, satisfying the following conditions:*

(1) *for every* $i \in I$, *there exists* $q_i \in [0,1)$ *such that* $q_i \le a + \min\{b,c\}$ *and* $\rho(T_i(x), T_i(y)) \le q_i \rho(x,y)$ *for all* $x, y \in C_i$;
(2) *for every ordered pair* $i, j \in I$, $i \neq j$, *and all* $x \in C_i$, $y \in C_j$,

$$\rho(T_i(x), T_j(y)) \le a\rho(x,y) + b\rho(x, T_i(x)) + c\rho(y, T_j(y)). \tag{3.1}$$

Then T *is a Reich mapping with coefficients* a, b, c.

Proof. Let $x, y \in X$, and choose the unique $i, j \in I$ such that $x \in C_i$ and $y \in C_j$.

Suppose first that $i = j$. Put $D = \rho(T_i(x), T_i(y))$, $d_0 = \rho(x,y)$, $r_x = \rho(x, T_i(x))$, and $r_y = \rho(y, T_i(y))$. If $x = y$, then $D = 0$, so the required Reich inequality is immediate. Assume that $x \neq y$. Since $D \le q_i d_0$ and $q_i < 1$, we have $D < d_0$.

The ultrametric inequality applied successively to the chain $x, T_i(x), T_i(y), y$ gives
$d_0 \le \max\{r_x, D, r_y\}$. The term D is strictly smaller than d_0, and therefore it cannot be the only term realizing the maximum. Consequently, $d_0 \le \max\{r_x, r_y\}$.

If $q_i \le a$, then $D \le ad_0$, and the desired estimate follows because the residual terms are nonnegative. Suppose that $q_i > a$. From $q_i \le a + \min\{b,c\}$, we have $q_i - a \le \min\{b,c\}$. Hence

$$\begin{aligned} D &\le q_i d_0 = ad_0 + (q_i - a)d_0 \\ &\le ad_0 + \min\{b,c\} \max\{r_x, r_y\} \\ &\le ad_0 + br_x + cr_y. \end{aligned}$$

Indeed, if $r_x \ge r_y$, then $\min\{b,c\}\max\{r_x, r_y\} \le br_x$; if $r_y \ge r_x$, then it is at most cr_y. Thus the Reich inequality holds when x and y belong to the same branch.

Suppose now that $i \neq j$. Since $T(x) = T_i(x)$ and $T(y) = T_j(y)$, condition (3.1) gives directly

$$\rho(T(x), T(y)) \le a\rho(x,y) + b\rho(x, T(x)) + c\rho(y, T(y)).$$

Therefore the Reich inequality holds for every ordered pair $x, y \in X$, and T is a Reich mapping. □

We next pass to first-disagreement depths. Put $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$, with the convention $2^{-\infty} = 0$.

Definition 3.5. Let $a, b, c \ge 0$ satisfy $a + b + c < 1$, and let $i, j \in I$, $i \neq j$. A pair $(h_{ij}, \mathcal{D}_{ij})$, where $h_{ij} \in \mathbb{N}_0$ and $\mathcal{D}_{ij} \subseteq \overline{\mathbb{N}}_0^3$ is finite and nonempty, is called directed Reich depth-pattern data for the ordered branch pair $(\mathcal{C}_i, \mathcal{C}_j)$ if:

(1) for every $A \in \mathcal{C}_i$, $B \in \mathcal{C}_j$, and $w \in \Sigma^*$ with $|w| < h_{ij}$, one has $\mathbf{1}_{T_i(A)}(w) = \mathbf{1}_{T_j(B)}(w)$;
(2) for every $A \in \mathcal{C}_i$ and $B \in \mathcal{C}_j$, there exists $(\ell, r, s) \in \mathcal{D}_{ij}$ such that $d(A, B) \geq 2^{-\ell}$, $d(A, T_i(A)) \geq 2^{-r}$, and $d(B, T_j(B)) \geq 2^{-s}$;
(3) every $(\ell, r, s) \in \mathcal{D}_{ij}$ satisfies $2^{-h_{ij}} \leq a2^{-\ell} + b2^{-r} + c2^{-s}$.

The family $\mathcal{D}_{ij}$ permits different pairs of languages to be controlled by different combinations of the input distance and the two residual displacements. The data are directed because, when $b \neq c$, interchanging the two languages also interchanges the roles of their residuals.

Theorem 3.6. *Let $T : \mathcal{L} \to \mathcal{L}$ be the operator from Definition 3.1. Suppose that there exist $a, b, c \geq 0$, $a + b + c < 1$, such that $\kappa_i \leq a + \min\{b, c\}$ for every $i \in I$, and every ordered pair $(\mathcal{C}_i, \mathcal{C}_j)$, $i \neq j$, admits directed Reich depth-pattern data. Then, for all $A, B \in \mathcal{L}$,*

$$d(T(A), T(B)) \leq a\, d(A, B) + b\, d(A, T(A)) + c\, d(B, T(B)). \tag{3.2}$$

Consequently, T is a Reich mapping.

Proof. We verify the assumptions of Proposition 3.4. By Proposition 3.2, every branch operator satisfies $d(T_i(A), T_i(B)) \leq \kappa_i d(A, B)$. Since $\kappa_i \leq a + \min\{b, c\} < 1$, the first assumption of Proposition 3.4 holds with $q_i = \kappa_i$.

Let $i \neq j$, $A \in \mathcal{C}_i$, and $B \in \mathcal{C}_j$. The first condition in Definition 3.5 shows that $T_i(A)$ and $T_j(B)$ agree on all words of length strictly less than h_{ij}. By Theorem 2.2, $d(T_i(A), T_j(B)) \leq 2^{-h_{ij}}$.

Choose a pattern $(\ell, r, s) \in \mathcal{D}_{ij}$ supplied by the second condition. Then $d(A, B) \geq 2^{-\ell}$, $d(A, T_i(A)) \geq 2^{-r}$, and $d(B, T_j(B)) \geq 2^{-s}$. Using the numerical condition for this pattern, we obtain

$$\begin{aligned} d(T_i(A), T_j(B)) &\leq 2^{-h_{ij}} \\ &\leq a2^{-\ell} + b2^{-r} + c2^{-s} \\ &\leq a\, d(A, B) + b\, d(A, T_i(A)) + c\, d(B, T_j(B)). \end{aligned}$$

This is exactly the required cross-branch estimate. Proposition 3.4 therefore applies and gives (3.2). □

We first record a general consequence of the strong triangle inequality.

Proposition 3.7. *Let (X, ρ) be an ultrametric space, and let $T : X \to X$ be a Reich mapping with coefficients $a, b, c \geq 0$, $a + b + c < 1$. Put $q = a + b + c$. Then*

$$\rho(Tx, Ty) \leq q \max\{\rho(x, Tx), \rho(y, Ty)\} \tag{3.3}$$

for all $x, y \in X$. Consequently, every Reich mapping on an ultrametric space is a Bianchini contraction.

Proof. Let $D = \rho(Tx, Ty)$, $R = \max\{\rho(x, Tx), \rho(y, Ty)\}$, $d_0 = \rho(x, y)$. By the ultrametric inequality applied to the chain x, Tx, Ty, y, $d_0 \leq \max\{R, D\}$.

Suppose first that $D \leq R$. Then $d_0 \leq R$, and the Reich inequality gives $D \leq aR + (b + c)R = qR$.

Suppose now that $D > R$. Then $d_0 \leq D$, and hence $D \leq aD + (b + c)R$. Since $a < 1$, $D \leq \frac{b+c}{1-a}R$. Moreover, $q - \frac{b+c}{1-a} = \frac{a(1-q)}{1-a} \geq 0$. Therefore $D \leq qR$, which proves (3.3). □

The dyadic range of the length-based language metric yields the converse implication.

Theorem 3.8. *For a mapping $T : \mathcal{L} \to \mathcal{L}$, Reich contractivity and Bianchini contractivity are equivalent.*

More precisely, every Reich mapping on $(\mathcal{L}, d)$ satisfies

$$d(T(A), T(B)) \leq \frac{1}{2} \max\{d(A, T(A)), d(B, T(B))\} \tag{3.4}$$

for all $A, B \in \mathcal{L}$. Conversely, every Bianchini contraction on $(\mathcal{L}, d)$ satisfies

$$d(T(A), T(B)) \leq \frac{1}{4}d(A, B) + \frac{1}{4}d(A, T(A)) + \frac{1}{4}d(B, T(B)) \tag{3.5}$$

for all $A, B \in \mathcal{L}$, and hence is a Reich mapping with coefficients $a = b = c = 1/4$. The factor $1/2$ in (3.4) and the common coefficient $1/4$ in (3.5) are sharp in general; see Example 4.1.

Proof. Suppose first that T is a Reich mapping. By Proposition 3.7, there exists $q < 1$ such that $d(T(A), T(B)) \leq q \max\{d(A, T(A)), d(B, T(B))\}$. Put $D = d(T(A), T(B))$, $R = \max\{d(A, T(A)), d(B, T(B))\}$. If $R = 0$, then $D = 0$. Assume that $R > 0$. If $D > 0$, both D and R are powers of 2. Since $D \leq qR < R$, their corresponding exponents differ by at least one, and therefore $D \leq R/2$. Thus (3.4) holds.

Conversely, suppose that T is a Bianchini contraction. The same dyadic argument shows that (3.4) holds. Set $D = d(T(A), T(B))$, $R_A = d(A, T(A))$, $R_B = d(B, T(B))$, $R = \max\{R_A, R_B\}$. If $R = 0$, then $D = 0$, and (3.5) is immediate. Assume $R > 0$, and suppose without loss of generality that $R_A = R$. By the ultrametric inequality applied to the chain $A, B, T(B), T(A)$, $R \leq \max\{d(A, B), R_B, D\}$. Since $D \leq R/2 < R$, at least one of $d(A, B)$ and R_B is at least R. Hence $d(A, B) + R_A + R_B \geq 2R$. Using $D \leq R/2$, we obtain $D \leq \frac{1}{2}R \leq \frac{1}{4}\big(d(A, B) + R_A + R_B\big)$, which is precisely (3.5). Since $1/4 + 1/4 + 1/4 = 3/4 < 1$, T is a Reich mapping. □

The structural equivalence above does not make the two verification mechanisms identical. The directed Reich criterion permits asymmetric combinations of the input distance and the two residual displacements, whereas a direct Bianchini criterion uses only their maximum. The following result extends the Kannan max-type criterion of [2] to the critical dyadic coefficient 1/2, which remains admissible for Bianchini contractivity but is excluded by the classical Kannan condition.

Theorem 3.9. *Assume that $\kappa_i < 1$ for every $i \in I$. Suppose that, for each pair $i < j$, there exist integers $0 \le r_{ij} < h_{ij}$ with $h_{ij} - r_{ij} \ge 1$ such that, for every $A \in \mathcal{C}_i$ and $B \in \mathcal{C}_j$,*

(1) $T_i(A)$ and $T_j(B)$ agree on every word of length strictly less than h_{ij};
(2) $\max\{d(A, T_i(A)), d(B, T_j(B))\} \ge 2^{-r_{ij}}$.

Define $\lambda = \max\left\{\max_{i\in I} \kappa_i,\ \max_{i<j} 2^{-(h_{ij}-r_{ij})}\right\}$.
Then $0 \le \lambda \le 1/2 < 1$, and

$$d(T(A), T(B)) \le \lambda \max\{d(A, T(A)), d(B, T(B))\} \tag{3.6}$$

for all $A, B \in \mathcal{L}$. Consequently, T is a Bianchini contraction and, by Theorem 3.8, also a Reich mapping. It has a unique fixed language $L^* \in \mathcal{L}$, and the Picard iteration converges to L^* from every initial language.

Proof. If $p_i = 0$, then $\kappa_i = 0$. If $p_i \ge 1$, then $\kappa_i = 2^{-m_i} < 1$. Since m_i is a nonnegative integer, this implies $m_i \ge 1$, and hence $\kappa_i \le 1/2$. Moreover, $h_{ij} - r_{ij} \ge 1$ gives $2^{-(h_{ij}-r_{ij})} \le \frac{1}{2}$. Therefore $0 \le \lambda \le 1/2 < 1$.

Let $A, B \in \mathcal{L}$. Suppose first that $A, B \in \mathcal{C}_i$. Put $D = d(T_i(A), T_i(B))$. If $A = B$, then $D = 0$. Assume that $A \ne B$. By Proposition 3.2, $D \le \kappa_i d(A, B) < d(A, B)$. Applying the ultrametric inequality along the chain A, $T_i(A)$, $T_i(B)$, B, we obtain $d(A, B) \le \max\{d(A, T_i(A)), D, d(B, T_i(B))\}$. Since $D < d(A, B)$, it follows that $d(A, B) \le \max\{d(A, T_i(A)), d(B, T_i(B))\}$. Therefore, $D \le \lambda \max\{d(A, T(A)), d(B, T(B))\}$.

Suppose now that $A \in \mathcal{C}_i$ and $B \in \mathcal{C}_j$, where $i \ne j$. Put $i_0 = \min\{i, j\}$ and $j_0 = \max\{i, j\}$. If $i < j$, the required estimates follow directly from the assumptions. If $i > j$, they follow by applying the assumptions for the pair (j, i) to $B \in \mathcal{C}_j$ and $A \in \mathcal{C}_i$, and then using the symmetry of the metric and of the maximum. Hence $d(T(A), T(B)) \le 2^{-h_{i_0j_0}}$, and $\max\{d(A, T(A)), d(B, T(B))\} \ge 2^{-r_{i_0j_0}}$. Hence

$$\begin{aligned} d(T(A), T(B)) &\le 2^{-h_{i_0j_0}} \\ &= 2^{-(h_{i_0j_0}-r_{i_0j_0})} 2^{-r_{i_0j_0}} \\ &\le \lambda \max\{d(A, T(A)), d(B, T(B))\}. \end{aligned}$$

Thus (3.6) holds for all $A, B \in \mathcal{L}$. By Definition 2.6, T is a Bianchini contraction. Since $(\mathcal{L}, d)$ is complete, Theorem 2.7 gives the fixed-point and Picard conclusions. □

The Bianchini max-residual estimate yields an exact a posteriori error identity in arbitrary ultrametric spaces.

Proposition 3.10. *Let (X, ρ) be an ultrametric space, let $T : X \to X$ be a Bianchini contraction with coefficient $\lambda \in [0, 1)$, and let $x^* \in X$ be a fixed point of T. Then, for every $x \in X$,*

$$\rho(x, x^*) = \rho(x, T(x)). \tag{3.7}$$

Moreover, $\rho(T(x), x^*) \leq \lambda \rho(x, T(x))$.

Proof. Put $R = \rho(x, T(x))$ and $E = \rho(x, x^*)$. Since $T(x^*) = x^*$, the Bianchini inequality gives $\rho(T(x), x^*) = \rho(T(x), T(x^*)) \leq \lambda R$. The ultrametric inequality therefore yields $E \leq \max\{R, \lambda R\} = R$.

Conversely, $R \leq \max\{E, \rho(x^*, T(x))\} \leq \max\{E, \lambda R\}$. If $R > E$, then both terms on the right are strictly smaller than R, which is impossible. Hence $R \leq E$. Therefore $E = R$, proving (3.7). The second assertion was obtained above. □

Remark 3.11. Proposition 3.10 does not require completeness of the ultrametric space. Completeness is needed only when the existence of a fixed point is to be obtained from the Bianchini fixed-point theorem. Once a fixed point exists, the exact residual identity follows solely from the Bianchini inequality and the strong triangle inequality.

Corollary 3.12. *Let $T : \mathcal{L} \to \mathcal{L}$ be a Bianchini contraction, let L^* be its fixed language, and let $L \in \mathcal{L}$. If $d(L, T(L)) = 2^{-k}$ for some $k \in \mathbb{N}_0$, then L and L^* agree on every word of length strictly less than k and disagree on at least one word of length k. In particular, if $d(L, T(L)) \leq 2^{-N}$, then L and L^* agree on every word of length strictly less than N.*

Proof. By Proposition 3.10, $d(L, L^*) = d(L, T(L))$. The conclusion follows directly from Theorem 2.2. □

Geometric Picard estimates for Bianchini contractions are available in broader semimetric settings, including ultrametric specializations [6]. In the ultrametric setting considered here, Proposition 3.10 sharpens this convergence information by identifying the fixed-point error exactly with the current Picard displacement.

Proposition 3.13. *Let (X, ρ) be an ultrametric space, let $T : X \to X$ be a Bianchini contraction with coefficient $\lambda \in [0, 1)$, and let $x^* \in X$ be a fixed point of T. For $x_0 \in X$, define $x_{n+1} = T(x_n)$ and put*

$$D_n = \rho(x_n, x_{n+1}).$$

Then, for every $n \geq 0$, $D_n \leq \lambda^n D_0$ and $\rho(x_n, x^*) = D_n \leq \lambda^n D_0$.

Proof. For every $n \geq 1$, applying the Bianchini inequality to x_{n-1} and x_n gives $D_n \leq \lambda \max\{D_{n-1}, D_n\}$. If $D_n > D_{n-1}$, then the right-hand side equals $\lambda D_n < D_n$, a contradiction. Hence $D_n \leq D_{n-1}$, and therefore $D_n \leq \lambda D_{n-1}$. Induction yields $D_n \leq \lambda^n D_0$. Finally, Proposition 3.10, applied to x_n, gives

$$\rho(x_n, x^*) = \rho(x_n, T(x_n)) = D_n.$$

□

Corollary 3.14. *Let $T : \mathcal{L} \to \mathcal{L}$ be a Bianchini contraction with coefficient $\lambda \in [0, 1)$, let L^* be its fixed language, and let $L^{(n+1)} = T(L^{(n)})$. Put $D_0 = d(L^{(0)}, L^{(1)})$. If $N, n \in \mathbb{N}_0$ satisfy $\lambda^n D_0 \leq 2^{-N}$, then $L^{(n)}$ and L^* agree on every word of length strictly less than N.*

Proof. By Proposition 3.13, $d(L^{(n)}, L^*) \leq \lambda^n D_0 \leq 2^{-N}$. The conclusion follows from Theorem 2.2. □

The Bianchini criterion contains the previously established Kannan regime for finite-observation conditional guarded language operators as the stricter case in which the max-residual coefficient is smaller than 1/2 [2].

Corollary 3.15 ([12, 2]). *Assume that $\kappa_i < 1/2$ for every $i \in I$. Suppose that, for each pair $i < j$, there exist integers $0 \leq r_{ij} < h_{ij}$ with $h_{ij} - r_{ij} \geq 2$ such that, for every $A \in \mathcal{C}_i$ and $B \in \mathcal{C}_j$,*

(1) $T_i(A)$ and $T_j(B)$ agree on every word of length strictly less than h_{ij};
(2) $\max\{d(A, T_i(A)), d(B, T_j(B))\} \geq 2^{-r_{ij}}$.

Define $\alpha = \max\left\{\max_{i\in I} \kappa_i,\ \max_{i<j} 2^{-(h_{ij}-r_{ij})}\right\}$.
Then $\alpha < 1/2$, and

$$d(T(A), T(B)) \leq \alpha \max\{d(A, T(A)), d(B, T(B))\} \tag{3.8}$$

for all $A, B \in \mathcal{L}$. Consequently, $d(T(A), T(B)) \leq \alpha\big(d(A, T(A)) + d(B, T(B))\big)$, so T is a Kannan mapping. In particular, T is also a Reich mapping with coefficients $a = 0$ and $b = c = \alpha$.

Proof. The assumptions of Theorem 3.9 are satisfied, and its coefficient λ is precisely α. Moreover, $\kappa_i < 1/2$ for every i, while $h_{ij} - r_{ij} \geq 2$ gives $2^{-(h_{ij}-r_{ij})} \leq \frac{1}{4} < \frac{1}{2}$. Hence $\alpha < 1/2$. Theorem 3.9 yields (3.8). Since

$$\max\{d(A,T(A)), d(B,T(B))\} \leq d(A,T(A)) + d(B,T(B)),$$

the classical Kannan inequality follows. Finally, $2\alpha < 1$, so the same inequality is the Reich condition with $a = 0$ and $b = c = \alpha$. □

Remark 3.16. Theorem 3.8 and Corollary 3.15 clarify the relation between the three contractive regimes on the language space. Reich and Bianchini contractivity are equivalent, although their defining inequalities have different forms. By contrast, the classical Kannan condition requires a coefficient strictly smaller than 1/2. The direct depth–residual Bianchini criterion may attain the critical dyadic value $\lambda = 1/2$, at which the max-residual estimate no longer yields the classical Kannan condition with the same coefficient. Example 4.1 shows that this distinction is genuine: the value 1/2 may be the optimal Bianchini coefficient while the operator fails to be Kannan-contractive.

The image-agreement condition in Definition 3.5 can be checked directly from the seed languages and guard lengths.

Proposition 3.17. *Let $i, j \in I$, $i \neq j$, and let $h_{ij} \in \mathbb{N}_0$. Suppose that S_i and S_j agree on every word of length strictly less than h_{ij}. Suppose also that, for each $k \in \{i, j\}$, either $p_k = 0$ or $m_k \geq h_{ij}$. Then $T_i(A)$ and $T_j(B)$ agree on every word of length strictly less than h_{ij} for all $A \in \mathcal{C}_i$ and $B \in \mathcal{C}_j$.*

Proof. Take $A \in \mathcal{C}_i$, $B \in \mathcal{C}_j$, and $w \in \Sigma^*$ with $|w| < h_{ij}$. If $p_i = 0$, then $F_i(A) = \varnothing$. If $p_i \geq 1$, every word in $F_i(A)$ has the form $u_{i,r} x v_{i,r}$, where $x \in A$, and therefore has length at least $|u_{i,r}| + |v_{i,r}| \geq m_i \geq h_{ij}$. Thus $w \notin F_i(A)$, and consequently $\mathbf{1}_{T_i(A)}(w) = \mathbf{1}_{S_i}(w)$.

The same argument gives $\mathbf{1}_{T_j(B)}(w) = \mathbf{1}_{S_j}(w)$. Since S_i and S_j agree below h_{ij}, $\mathbf{1}_{T_i(A)}(w) = \mathbf{1}_{S_i}(w) = \mathbf{1}_{S_j}(w) = \mathbf{1}_{T_j(B)}(w)$. Since w was arbitrary, the conclusion follows. □

Corollary 3.18. *Under the assumptions of Theorem 3.6, the operator T has a unique fixed language $L^* \in \mathcal{L}$, and the Picard iteration $L^{(n+1)} = T(L^{(n)})$ converges to L^* for every $L^{(0)} \in \mathcal{L}$.*

Proof. The space $(\mathcal{L}, d)$ is complete by Theorem 2.2, while Theorem 3.6 shows that T is a Reich mapping. The conclusion follows from Theorem 2.9. □

The structural equivalence with Bianchini contractivity yields a uniform dyadic Picard estimate for the Reich regime on the language space.

Proposition 3.19. *Under the assumptions of Theorem 3.6, let L^* be the unique fixed language and let $L^{(n+1)} = T(L^{(n)})$. Put $D_n = d(L^{(n)}, L^{(n+1)})$. Then, for every $L \in \mathcal{L}$, $d(L, L^*) = d(L, T(L))$, and, for every $n \geq 0$,*

$$d(L^{(n)}, L^*) = D_n \leq 2^{-n} D_0.$$

Proof. By Theorem 3.6, T is a Reich mapping. Theorem 3.8 therefore gives the Bianchini estimate $d(T(A), T(B)) \leq \frac{1}{2} \max\{d(A, T(A)), d(B, T(B))\}$. The exact residual identity follows from Proposition 3.10, while Proposition 3.13, applied with $\lambda = 1/2$, gives $d(L^{(n)}, L^*) = D_n \leq 2^{-n} D_0$. □

Corollary 3.20. *Under the assumptions and notation of Proposition 3.19, let $N, n \in \mathbb{N}_0$. If $2^{-n} D_0 \leq 2^{-N}$, then $L^{(n)}$ and L^* agree on every word of length strictly less than N.*

Proof. By Proposition 3.19, $d(L^{(n)}, L^*) \leq 2^{-n} D_0 \leq 2^{-N}$. The conclusion follows from Theorem 2.2. □

4. An Illustrative Example

Example 4.1. Let $\Sigma = \{x\}$, put $z = x^5$, and take $W = \{x, z\}$. Define $\varphi : \{0,1\}^2 \to \{1,2\}$ by $\varphi(1,1) = 2$ and $\varphi(\eta) = 1$ for every $\eta \neq (1,1)$. Hence $\mathcal{C}_2 = \{L : x, z \in L\}$ and $\mathcal{C}_1 = \mathcal{L} \setminus \mathcal{C}_2$. Let $S_1 = \varnothing$, $S_2 = \{x^2\}$, and $G_1 = G_2 = \{(x^3, \varepsilon)\}$. The corresponding conditional operator is

$$T(L) = \begin{cases} x^3 L, & L \in \mathcal{C}_1, \\ \{x^2\} \cup x^3 L, & L \in \mathcal{C}_2. \end{cases} \tag{4.1}$$

Both branch coefficients are $\kappa_1 = \kappa_2 = 2^{-3} = 1/8$.

We first verify the Bianchini criterion. The two branch images contain neither ε nor x, so they agree on every word of length strictly less than 2. Thus we may take $h_{12} = 2$. Let $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$. Since $x \in B$ but $x \notin T_2(B)$, one has $d(B, T_2(B)) \geq 2^{-1}$. Hence the residual condition of Theorem 3.9 holds with $r_{12} = 1$. Since $h_{12} - r_{12} = 1$, the corresponding Bianchini coefficient is $\lambda = \max\left\{\frac{1}{8}, 2^{-(2-1)}\right\} = \frac{1}{2} < 1$. Therefore

$$d(T(A), T(B)) \leq \frac{1}{2} \max\{d(A, T(A)), d(B, T(B))\} \tag{4.2}$$

for all $A, B \in \mathcal{L}$. Thus T is a Bianchini contraction.

The operator also satisfies the general Reich criterion. Choose $a = 1/3$, $b = 1/6$, and $c = 1/3$. Then $a+b+c = 5/6 < 1$ and $\kappa_i = \frac{1}{8} \leq a + \min\{b, c\} = \frac{1}{2}$. For $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$, if $x \notin A$, then $d(A, B) \geq 2^{-1}$, while if $x \in A$, then $x \notin T_1(A) = x^3 A$, and hence $d(A, T_1(A)) \geq 2^{-1}$. Together with $d(B, T_2(B)) \geq 2^{-1}$, this allows $\mathcal{D}_{12} = \{(1, \infty, 1), (\infty, 1, 1)\}$, $\mathcal{D}_{21} = \{(1, 1, \infty), (\infty, 1, 1)\}$. The numerical conditions follow from $2^{-2} = \frac{1}{4}$, $(a+c)2^{-1} = \frac{1}{3}$, $(a+b)2^{-1} = (b+c)2^{-1} = \frac{1}{4}$. Hence Theorem 3.6 also shows that T is a Reich mapping.

The operator is not a Banach contraction. Indeed, for $A = \{x\}$ and $B = \{x, x^5\}$, one has $d(A,B) = 2^{-5}$, $d(T(A),T(B)) = 2^{-2}$, so no Banach constant smaller than one can satisfy the contraction inequality.

More importantly, T is not a Kannan mapping. Take $A = \varnothing$ and $B = \{x, x^5\}$. Then $T(A) = \varnothing$, $d(T(A),T(B)) = \frac{1}{4}$, while $d(A,T(A)) = 0$, $d(B,T(B)) = \frac{1}{2}$. A classical Kannan inequality would therefore require $\frac{1}{4} \leq \alpha\left(0 + \frac{1}{2}\right)$, and hence $\alpha \geq 1/2$, which is excluded in the classical Kannan condition. Thus the operator is Bianchini-contractive but neither Banach-contractive nor Kannan-contractive.

Moreover, the same pair shows that the Bianchini coefficient $\lambda = 1/2$ is optimal for this operator, since $d(T(A),T(B)) = \frac{1}{2}\max\{d(A,T(A)), d(B,T(B))\}$. Thus the example realizes the critical dyadic value $1/2$: the Bianchini estimate remains valid with an optimal coefficient, whereas the classical Kannan condition fails.

The same pair also proves that the common coefficient $1/4$ in the universal symmetric Reich estimate (3.5) is sharp. Indeed, $d(A,B) = \frac{1}{2}$, $d(A,T(A)) = 0$, $d(B,T(B)) = \frac{1}{2}$, and $d(T(A),T(B)) = \frac{1}{4} = \frac{1}{4}\big(d(A,B) + d(A,T(A)) + d(B,T(B))\big)$. Thus no smaller common coefficient can replace $1/4$ in (3.5) for all Bianchini contractions on $(\mathcal{L}, d)$.

Finally, $T(\varnothing) = \varnothing$, so the unique fixed language is $L^* = \varnothing$. Proposition 3.10 gives the exact a posteriori identity $d(L,\varnothing) = d(L,T(L))$ for every $L \in \mathcal{L}$. Moreover, Proposition 3.13 yields $d(L^{(n)},\varnothing) = D_n \leq 2^{-n}D_0$. The same rate is also available from the Reich classification through Theorem 3.8 and Proposition 3.19. Thus the example simultaneously exhibits the sharp universal Bianchini factor $1/2$, the sharp common coefficient $1/4$ in the universal symmetric Reich estimate, failure of both Banach and Kannan contractivity, and a directed Reich representation with the asymmetric coefficients $a = 1/3$, $b = 1/6$, and $c = 1/3$.

5. Conclusions

We have developed Reich and Bianchini fixed-point criteria for conditional guarded language operators based on finite observations, acting on the language space equipped with the length-based first-disagreement ultrametric. The directed Reich depth-pattern condition permits asymmetric use of the input distance and the two residual displacements, yielding global Reich contractivity even when branch switching destroys Banach contractivity.

Beyond the conditional language construction, we have shown that every Reich mapping on an ultrametric space is a Bianchini contraction. On the dyadic language space, the converse also holds, so Reich and Bianchini contractivity are equivalent. The dyadic structure further yields the universal max-residual coefficient $1/2$. For Bianchini contractions with fixed points on ultrametric spaces, the residual gives the exact distance to the fixed point, and the Picard displacement coincides exactly with the current fixed-point error, sharpening the usual geometric convergence information.

For finite-observation conditional guarded operators, the depth–residual criterion reaches the critical dyadic value $1/2$. The previously established Kannan criterion is recovered under the stricter threshold below $1/2$, while the illustrative example shows that the value $1/2$ may be optimal and that Bianchini and Reich contractivity may hold even when both Banach and classical Kannan contractivity fail. In the language setting, the resulting residual and Picard estimates translate directly into finite-depth correctness certificates.

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СВИВАНИЯ ОТ ТИП REICH И BIANCHINI ЗА УСЛОВНИ ОПЕРАТОРИ С ГАРДОВЕ И КРАЕН НАБОР ОТ НАБЛЮДЕНИЯ ВЪРХУ ФОРМАЛНИ ЕЗИЦИ

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Резюме. Изследваме условни оператори с гардове и краен набор от наблюдения върху пространството на всички формални езици над крайна азбука, снабдено с ултраметрика, определена от дължината на най-късата различаваща се дума. Установяваме общ насочен критерий, основан на дълбочината, който гарантира, че такъв оператор удовлетворява неравенството на Reich с произволни коефициенти $a, b, c \geq 0$, за които $a + b + c < 1$. След това показваме, че свиванията от тип Reich и Bianchini са еквивалентни в пространството на формалните езици: всеки оператор на Reich в ултраметричното пространство е оператор на Bianchini, а от диадичната структура на метриката върху езиците следва, че всеки оператор на Bianchini удовлетворява неравенство на Reich с коефициенти $a = b = c = 1/4$. За условните оператори с гардове получаваме критерий, формулиран чрез дълбочина и остатъчно разстояние, който достига критичния диадичен коефициент $1/2$, недопустим в класическото условие на Kannan. Получаваме също точно апостериорно тъждество за остатъчното разстояние при свивания от тип Bianchini с неподвижна точка в ултраметрични пространства. Това тъждество показва, че разстоянието до неподвижната точка съвпада точно с остатъчното разстояние, и води до точни тъждества между грешката и преместването при итерациите на Picard, както и до удостоверяване на коректността до зададена крайна дълбочина в пространството на формалните езици. По-рано установеният критерий на Kannan се получава при по-строгото изискване коефициентът да бъде строго по-малък от $1/2$.