RELATIONSHIP BETWEEN PONCELET THEOREM AND EULER FORMULAE FOR DISTANCES IN THE TRIANGLE

Sava Grozdev, Veselin Nenkov, Tatiana Madjarova

Abstract. *It is shown the construction of a triangle when its circum-radius and in-radius are known. A justification is proposed by a remarkable theorem of the French mathematician Poncelet and Euler formulae for the distances between the circum-circle centre and its in-circles centres. The described construction is applied to some loci when the triangle is moving remaining inscribed in a circle and circumscribed with respect to a second circle.*

Key words: triangle, circumcircle, incircle, excircle, Poncelet theorem, Euler circle.

Since each triangle ABC has a circumcircle Γ and an incircle ω, the following two basic questions arise:

1) How circles Γ and ω, with radii R and r, should be located in a plane that one triangle ABC exists at least which is inscribed in Γ and circumscribed with respect to ω? In other words, how the centres O and J of Γ and ω, respectively, should be located that a triangle ABC exists which is inscribed in Γ and circumscribed with respect to ω.
2) If the circles Γ and ω are located in such a way that a triangle ABC exists, which is inscribed in Γ and circumscribed with respect to ω, how to determine the set of all triangles, which are inscribed in Γ and circumscribed with respect to ω?

The answer of the second question is given by the following

Theorem 0.1. ***Poncelet theorem***. *If the circles Γ and ω are located in the plane in such a way that a triangle ABC exists, which is inscribed in Γ and circumscribed with respect to ω, then each point on Γ is a vertex of a unique triangle, which is inscribed in Γ and circumscribed with respect to ω [12].*

To a certain extent, this theorem answers to the first question too in the following way (Fig. 1):

1) Construct an arbitrary circle Γ with centre O and radius R;
2) Using three points on Γ construct a triangle ABC;
3) Construct the in-circle ω in $\triangle ABC$;
4) Choose an arbitrary point A_1 on Γ and pass the tangent lines t_1 and t_2 to ω through A_1;
5) If t_1 and t_2 intersect Γ in the points B_1 and C_1, respectively, the triangle under search is $A_1B_1C_1$ (Fig. 1).

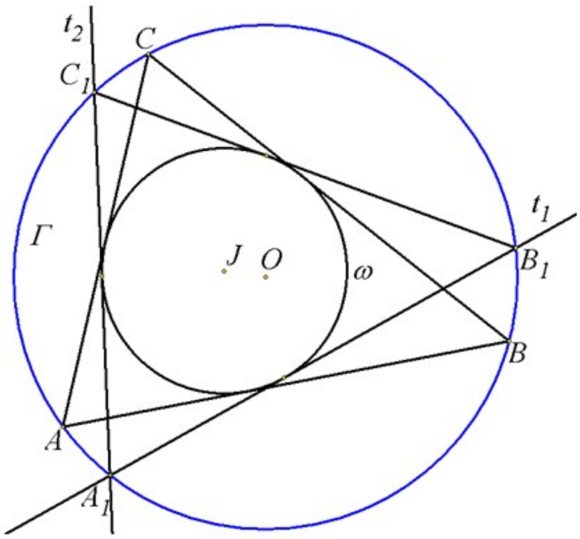

Figure 1.

The disadvantage of such a construction, aiming to answer to the first question, consists in ignoring the in advance-known radius r of the circle ω. This means that a certain correction is needed which gives the possibility of controlling the value of r. A solution is to use the following

Theorem 0.2. *If $\triangle ABC$ is inscribed in the circle $\Gamma(O, R)$ and circumscribed with respect to the circle $\omega(J, r)$, then the following equality is satisfied $OJ^2 = R^2 - 2Rr$ [9, 10, 12].*

The equality in question is known to be Euler formula. A direct consequence of theorem 0.2 is the following:

Corollary 0.1. *It is satisfied the inequality $R \geq 2r$ for each triangle ABC and the equality holds true only for the equilateral triangle.*

The corollary determines an upper bound for the radius r. This means that the construction of the circle ω is possible only if $r \leq \frac{R}{2}$.

The mentioned construction could be reworked in the following way using Euler formular from theorem 0.2 (Fig. 2):

1) Construct an arbitrary circle Γ with centre O and radius R;
2) Construct a circle k with centre O and radius $\sqrt{R^2 - 2Rr}$;
3) Choose an arbitrary point J on k;
4) Construct a circle ω with centre J and radius r;
5) Pass the tangent lines t_1 and t_2 to ω through an arbitrary point A on Γ;
6) If t_1 and t_2 intersect Γ in the points B and C, respectively, the triangle under search is ABC (Fig. 2).

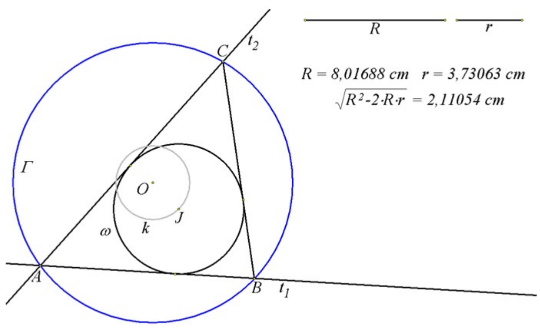

Figure 2.

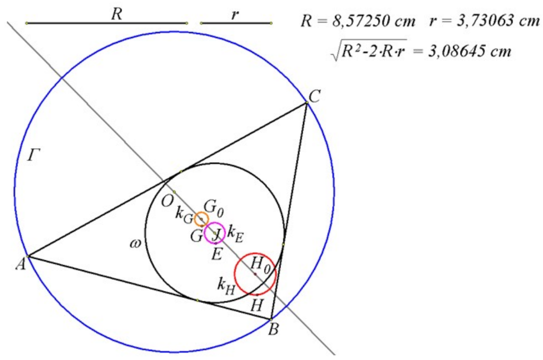

Figure 3.

It follows from this construction that in moving the point A on the circle Γ, we will obtain a triangle ABC always, which is inscribed in Γ and circumscribed with respect to ω. For this reason each notable point for he moving triangle ABC will describe a locus when ABC behaves in the considered way between the two fixed circles Γ and ω. Some of the loci are connected with the following assertions:

Theorem 0.3. *The orthocentre H of $\triangle ABC$ describes a circle k_H with centre H_0 on the line OJ and radius $R - 2r$ (Fig. 3).*

Theorem 0.4. *The gravity centre G of $\triangle ABC$ describes a circle k_G with centre G_0 on the line OJ and radius $\dfrac{R-2r}{3}$ (Fig. 3).*

Theorem 0.5. *The centre E of Euler circle describes a circle k_E with centre J and radiusc $\dfrac{R-2r}{2}$ (Fig. 3).*

The proofs and a generalization of these assertions are shown in [2].

The circles Γ and $\omega = \omega_a$ could be the circumcircle and an excircle, respectively. Then Poncelet theorem remains in power, while theorem 0.2 is changed in the following way:

Theorem 0.6. *If $\triangle ABC$ is inscribed in the circle $\Gamma(O, R)$ and escribed with respect to the circle $\omega_a(J_a, r_a)$, then $OJ_a^2 = R^2 + 2Rr_a$ [9, 10, 12].*

The equality in this theorem is known to be Euler formula too.

Using Euler formula from theorem 0.6, we can obtain the following construction:

1) Construct circle Γ with centre O and radius R;
2) Construct circle k_a with centre O and radius $\sqrt{R^2 + 2Rr_a}$;
3) Choose an arbitrary point J_a on k_a;
4) Construct circle ω_a with centre J_a and radius r_a;
5) Pass the tangent lines t_1 and t_2 to ω_a through an arbitrary point A on Γ;
6) If t_1 and t_2 intersect Γ in the points B and C, respectively, the triangle under search is ABC.

It follows from this construction that in moving the point A on the circle Γ, we will obtain a triangle ABC always, which is inscribed in Γ and escribed with respect to ω_a. For this reason each notable point for he moving triangle ABC will describe a locus when ABC behaves in the considered way between the two fixed circles Γ and ω_a. Some of the loci are connected with the following assertions:

Theorem 0.7. *The orthocentre H of $\triangle ABC$ describes an arc on a circle $k_H^{'}$ with centre $H_0^{'}$ on the line OJ and radius $R + 2r$ (Fig. 4).*

Theorem 0.8. *The centre of gravity G of $\triangle ABC$ describes an arc on a circle $k_G^{'}$ with centre $G_0^{'}$ on the line OJ and radius $\dfrac{R+2r}{3}$ (Fig. 4).*

Theorem 0.9. *The centre E of Euler circle describes an arc on a circle k_E with centre J_a and radius $\frac{R+2r}{2}$ (Fig. 4).*

Proofs of these theorems will be published in "Mathematics Plus" Journal.

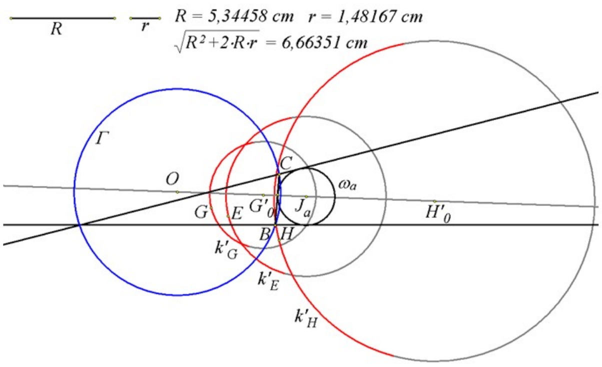

Figure 4.

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