THE CLASSES OF EQUIVALENCE IN THE SCHOOL COURSE IN MATHEMATICS

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ABSTRACT

The present paper offers a methodology for defining the number of solutions for some types of problems. The methodology is based on a unified approach to the classes of equivalence, caused by the relation of equivalence, defined in a certain set. The approach is applicable for the training of higher education students, future teachers in mathematics.

Key words: binary relations; classes of equivalence; number of solutions to the problem.

MOTIVATION, PROBLEMS AND SHORTCOMINGS

The idea for classes of equivalence is not popular in the school practice though during the last 30-40 years elements from the theory of relations have been introduced and are in use as means for solving problems of mathematical didactics in Bulgaria. A similar ascertainment is to be found with Kolmogorov A. N. (2010) too, according to whom the classes of equivalence remain known to mathematicians only and have not yet been introduced securely into the practice.

The knowledge of the shortcomings in the methodology for the use of the binary relations and the finding out of the causes for the problems provide the idea for a methodology for removal of such problems and shortcomings. A shortcoming relating to the study of the binary relations characteristics has been described in detail in Ganchev, Iv., Ninova, J. & Vutova, Ir., (2007) and the methodology for its removal has been proposed. Another shortcoming relating to the study of the binary relations, is the non-evident, not realized and not uniform use of the classes of equivalence when solving different problems. In some cases different representatives of one and the same class are identified and in other cases each class is identified with an arbitrary representative of its own, while in other cases

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the representatives of one and the same class are discussed as different elements. The replacement of one representative of a class of equivalence into another is expressed by different word formations (for example $\frac{4}{9}=\frac{8}{18}$ - extension of fraction, $\frac{7}{21}=\frac{1}{3}$ - cancellation of a fraction, $\frac{6}{\sqrt{2}}=\frac{6\sqrt{2}}{2}=3\sqrt{2}$ - rationalization of a denominator of a fraction, $3x+6x^2=3x(1+2x)$ - taking out a multiplier in front of the brackets, $3x(1+2x)=3x+6x^2$ - opening of brackets, $4{,}32=4{,}320$ - adding up zeros after the last meaningful digit,...). The general conclusion is that the approach to the classes of equivalence with different relations of equivalence is not unified.

During the recent years ideas of synergy are being introduced into the didactics of mathematics on an intended basis (Grozdev, C, 2003). Synergy is a science investigating into the joint action of a set of sub-systems as a result of which there arises a structure with a relevant functioning at a **macro**scopic level. In such systems a coordinated behavior of the sub-systems is to be observed as a result of which the degree of observing an arrangement rises. And the observation of good order provides on its part restriction of the chaos and the sporadic carrying out of the activities.

In order to enable such a good order and agreement with regard to the sporadic and unintended use of the classes of equivalence and of the differences (essential and terminological) a proposal is made of a methodology for overcoming, restricting, terminological unification and realized putting into practice of certain knowledge based on mathematical means, namely the relations of equivalence and the classes of equivalence. For the time being that is the purpose of training of the future teachers in mathematics, i.e. at a macro-level (Ninova, J., Ganchev, Iv., 2009).

THE METHODOLOGY

Let us discuss the following aggregate of problems, referring to different specific teaching contents.

Problem 1. Two points are given on the plane A and B. Construct a point C on the plane so that $\angle ACB=\gamma$, where γ is a given angle, and the distance from point C to straight line AB is equal to h_c.

Problem 2. Construct ΔABC, in the case $AB = c$, $\angle ACB = \gamma$ and the altitude to side AB is equal to h_c, where c and h_c are given segments, and γ is a given angle.

Problem 3. Construct ΔABC as per given $\frac{AB}{AC} = \frac{3}{4}$ and $\angle BAC = \alpha$, where α is a given angle.

Problem 4. Construct ΔABC as per given $\frac{AB}{AC} = \frac{3}{4}$, $\angle BAC = \alpha$ and median m_a to the side BC, where m_a is a given segment, and α is a given angle.

Problem 5. Solve the equation $\sin x = \frac{1}{2}$.

Problem 6. In the set of integers to solve the equation $5x + 12y = 13$.

Problem 7. To solve the equation $cos^2 x + cos^2 2x + cos^2 3x + cos^2 4x = 2$.

When solving problems for construction the number of solutions is defined in a different way. In the case of *non-position* problems the relation of equivalence represents the basis for defining the number of solutions to the problem. Since with such kind of problems the position of the figure representing the solution of the problem is not of importance, then each class of equivalence, obtained at a respective relation of equivalence, is identified with an arbitrary representative of its own. The solution of the problem is defined with accuracy to the kind of the respective relation of equivalence. All figures belonging to one class of equivalence are considered to represent one solution.

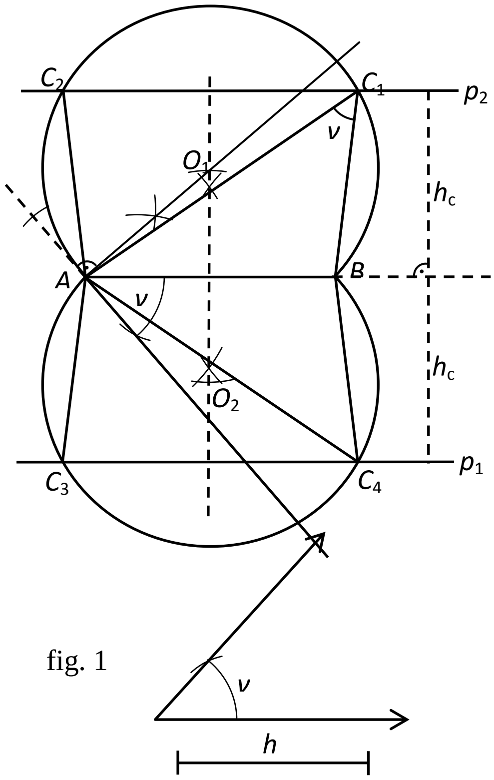

fig. 1

Different are the solutions belonging to different classes of equivalence. With the *position* problems all figures complying with the condition of the problem are considered as representing their solution.

The first problem discussed as a practical problem is a position problem.

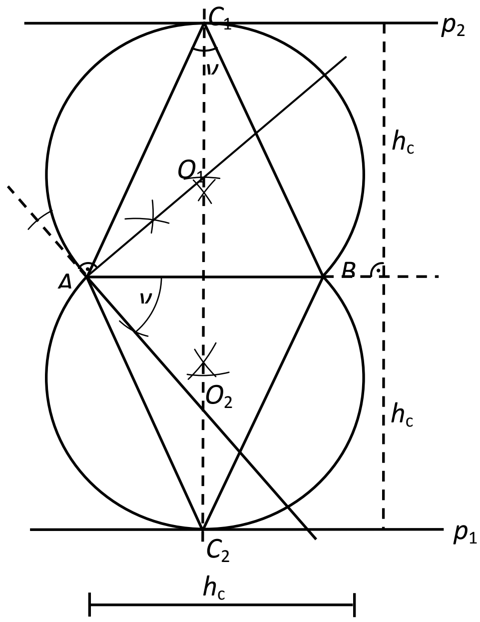

fig.2

It has respectively $4, 2$ or 0 solutions (fig.1, 2, 3) depending on which of the conditions for the given elements has been observed.

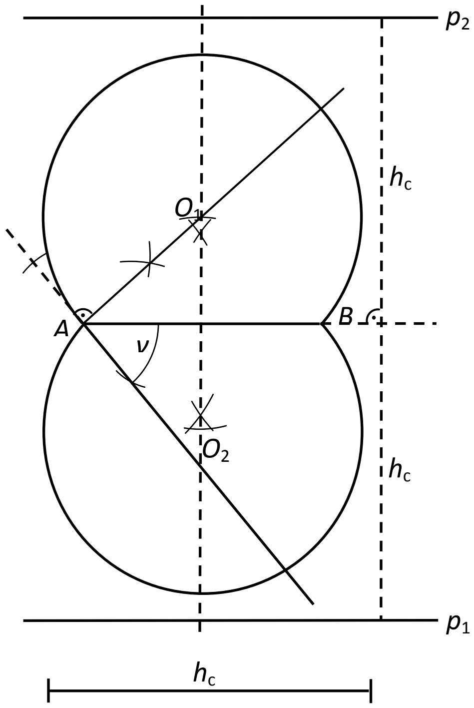

fig. 3 $\left(h_c \leq \frac{AB}{2} cot\, g \frac{\gamma}{2};\ h_c > \frac{AB}{2} cot\, g \frac{\gamma}{2} \right)$

The second problem is a non-position problem. The construction of the aimed triangle is executed following the same algorithm as with the first problem. But since the problem is a non-position one, then the equal triangles belong to one class of equivalence and therefore it is claimed that the problem has respectively 1 or 0 solutions defined **„*with an exactness up to equality*”** depending on the conditions $h_c \leq \frac{AB}{2} cot\, g \frac{\gamma}{2};\ \ h_c > \frac{AB}{2} cot\, g \frac{\gamma}{2}$.

There exists an arbitrary number of triangles satisfying the conditions of the problem 3. Therefore usually it is said at school that the problem has an indefinite number of solutions. All these triangles belong to one class of equivalence arising out of the relation „similarity", which is a relation of equivalence. If we follow the preliminary agreement of defining the number of solutions of non-position problems of construction, it has to be said that the problem has one solution defined **„*with an exactness up to similarity*".**

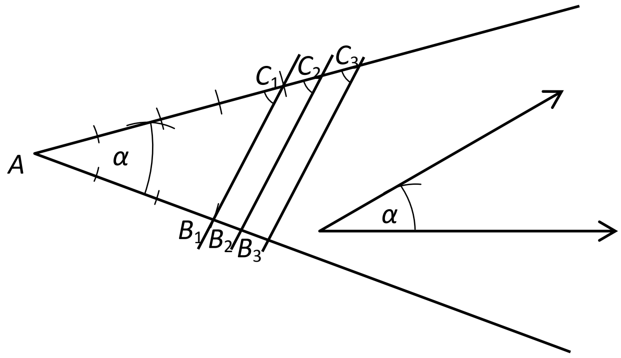

fig. 4

Out of that class of similar triangles by means of the linear element m_a it is possible to construct one triangle on the plane of the class of equal triangles. Therefore in this case it is said that the problem has one solution only defined **„*with exactness up to equality*"**.

That approach for defining the number of solutions to the problems is not typical only for the problems of construction. The relation „congruence as per a module..." is of importance when solving trigonometric equations (Vilenkin, N., Dunichev, K., Kaluzhnin, L. & Stoliar, A., 1980) and also of Diophantine equations.

As regards the solutions to the following problem it is said at school that the equation has an unlimited number of solutions that are recorded in the following way $- x = \frac{\pi}{6} + 2k\pi, k \in \square$ и $x = \frac{5\pi}{6} + 2k\pi, k \in \square$. Precisely, in the language of sets, the solutions to that equation may be recorded in the following way - $M_1 = \left\{ x \in \square \ / \ x = \frac{\pi}{6} + 2k\pi, k \in \square \right\}$, and $M_2 = \left\{ x \in \square \ / \ x = \frac{5\pi}{6} + 2k\pi, k \in \square \right\}$ or the solutions to the given equation are all real numbers $x \in M_1 \cup M_2$. The sets M_1 и M_2 may be recorded in the following way as well - $M_1 = \left\{ \ldots, -\frac{23\pi}{6}, -\frac{11\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}, \ldots \right\}$, $M_2 = \left\{ \ldots, -\frac{19\pi}{6}, -\frac{7\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{29\pi}{6}, \ldots \right\}$

The last shows that the sets M_1 and M_2 may be discussed as classes of equivalence obtained by breaking down the set of the real numbers from the relation ρ, defined in $\square$ in the following way - $x\rho y \leftrightarrow x \equiv y \pmod{2\pi}$. That relation is a relation of equivalence and in such case the sets M_1 and M_2 may be recorded in the following way - $M_1 = \left[\frac{\pi}{6}\right]_\rho$, and $M_2 = \left[\frac{5\pi}{6}\right]_\rho$. In such case as regards the number of solutions to the discussed trigonometric equation it may be said that in the interval $[0; 2\pi]$ the equation has two solutions defined by ***„with an exactness to congruence as per module*** 2π ***”.***

The solutions to the trigonometric equation $tgx = a, x \neq \frac{\pi}{2}(2k+1), k \in \mathbb{Z}$ are comparable as per module π. Therefore in the interval $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ that equation has one only solution.

According to a well known statement from the theory of the numbers the equation $5x + 12y = 13$ has an unlimited number of solutions, since $(5;12)/13$. The solutions to that equation are given by the formulae $\begin{cases} x = \alpha + 12t \\ y = \beta - 5t, \quad t \in \mathbb{Z} \end{cases}$. Therefore $x \in [\alpha]_{\rho_1}$, where ρ_1 is the relation „congruence as per module $\frac{b}{(a;b)}$”, defined in $\mathbb{Z}$. Therefore with an exactness to congruence as per module $\frac{b}{(a;b)}$ the first component x of the arranged couple $(x; y)$ has only one value, and therefore the second component y of the arranged couple has only one value. Therefore by the specified exactness the Diophantine equation (problem 6) has only one solution to $\mathbb{Z}$.

The Diophantine equations do not represent an objective of the training in mathematics at school. Still these represent means for defining the sets of non-crossing solutions in some trigonometric equations.

The equation to problem 7 is equivalent to the equation $cos\,x.cos\,2x.cos\,5x = 0$. A solution to the last equation is a union of sets of solutions to the equations $cos\,x = 0$, $cos\,2x = 0$ and $cos\,5x = 0$, which sets are respectively $M_1 = \left\{x \in \mathbb{R} \,/\, x = \frac{\pi}{2} + l\pi, l \in \mathbb{Z}\right\}$, $M_2 = \left\{x \in \mathbb{R} \,/\, x = \frac{\pi}{4} + \frac{5p\pi}{10}, p \in \mathbb{Z}\right\}$ и $M_3 = \left\{x \in \mathbb{R} \,/\, x = \frac{\pi}{10} + \frac{2k\pi}{10}, k \in \mathbb{Z}\right\}$, i.e. $x \in M_1 \cup M_2 \cup M_3$. Following the agreement that when recording the union of sets, the repeated elements are recorded once, a check is to be made if the sections of each two of the three sets obtained are not empty. That brings about the solving of the Diophantine equations $\frac{\pi}{10}+\frac{k\pi}{5}=\frac{\pi}{2}+l\pi$, $\frac{\pi}{10}+\frac{k\pi}{5}=\frac{\pi}{4}+\frac{p\pi}{2}$ and $\frac{\pi}{4}+\frac{p\pi}{2}=\frac{\pi}{2}+l\pi$. The last two equations have no solution. The first of these equations has a solution and when $k=5l+2$ solutions are obtained to the set M_1. Therefore $M_1 \subset M_3$ and then the solutions of the given equation are $x \in M_2 \cup M_3$. But since $M_2=\left[\frac{\pi}{4}\right]_{\rho_3}$, and $M_3=\left[\frac{\pi}{10}\right]_{\rho_3}$, where the relation ρ_3 has been defined in □ in the following way - $x\rho_3 y \leftrightarrow x \equiv y\left(\bmod \frac{\pi}{10}\right)$, then **„*with an exactness to congruence as per module*** $\frac{\pi}{10}$ **”** the discussed equation has two solutions in the interval $\left[0;\frac{\pi}{2}\right]$.

CONCLUSION

At the beginning we gave the grounds for the discussed problem by the ascertainment of A. N. Kolmogorov relating to the importance and the place of the relations of equivalence. In the quoted article (Kolmogorov A. N., 2010) the author proposes the use of these mathematical means for the solving of didactic problems relating to nearer terminological specifications. In the present paper we have shared some ideas regarding the use of the classes of equivalence for the solving of other terminological problems in the didactics of mathematics. The methodology of operation as proposed may be used as the basis for conscious systematization when practicing the contents of the school-books and school training aids. That is a criterion for the availability of a reflection by means of which the possibilities for application of the classes of equivalence are brought from a macro-level to the micro-level. Therefore these ideas are first of all subject of training of the students that are expected to be future teachers in mathematics. Thus it is expected that a complete equalization and agreement relating to that knowledge in the training of the school-children as well.

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