THE TRIAD OF ACTIVITIES SOLVING, FORMULATING AND TRANSFORMING OF MATHEMATICAL PROBLEMS

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ABSTRACT

By virtue of a survey of a number of publications dedicated to a separate study of the activities solving, formulating and transforming of mathematical problems, it is ascertained that in such cases the summary effect of all these activities is smaller than the effect, which is achieved when they are combined in a triad on the basis of the reflexive-synergetic approach. A conceptual model of this triad is elaborated.

The activity solving of mathematical problems is fundamental in the teaching of mathematics in secondary schools. In this sense, it is "prototypical" [1], while the formulation and transformation of problems are "derivative" activities deprived of meaning and significance if they are considered independently of the first one. So far in different sources these three activities have been the subject of independent examinations. Furthermore, there prevail publications, which regard mainly issues connected with the activity solving and methods of solving mathematical problems ([2], [3], [8], [19], [20], [22], [24], [28], [30], [32] etc.). During the last decade of the 20th century there were conducted specialized studies devoted to the activity of formulating mathematical problems ([31], [17], [29], [9], [10], [23], [33], etc.). Since the beginning of the 21st century, there have purposefully been investigated issues related to the activity of transforming mathematical problems ([11], [25], [27], [14], etc.). Meanwhile, some publications partially refer to issues connected with both the activity of solving mathematical problems, including teaching students how to solve problems, and the production of problems, for example investigating the problem of students' abilities for solving and formulating problems; for "developing" a certain problem; for formulating didactic systems of problems in accordance with predefined objectives ([21], [18], [12], [13], [26], [32], etc.).

In most of the sources quoted here, as well as in some other publications of ours, there have been investigated issues related to the essence of the activities of solving, formulating and transforming mathematical problems; possibilities for modeling some of their aspects; an analysis of these activities from various theoretical points of view. Furthermore, a large number of authors base their work fundamentally on the sources [4], [5], [6], [7], [30], etc, which explore some important contemporary ideas about improving the level of education with respect to solving mathematical problems.

Research work, however, shows that due to the existence of considerable correlations between the activities of solving, formulating and transforming mathematical problems, it is appropriate to set the problem of their complex treatment. Moreover, it is not a question of their mechanical unification, but of searching for possibilities to combine them on the basis of a system of approaches, in which the so called "reflexive-synergetic approach" [16] takes a crucial place. In the context of this approach, the activities formulating and transforming of problems can be used essentially in two aspects:

A) as "derivative" activities (in their capacity of heuristics) towards the main activity, which is the solving of problems;

B) as basic activities with a view to mastering of knowledge about theorems-signs and theorems-properties, basic problems, etc, and skills for their application on a cellular level as well as purposeful mastering of these heuristic activities.

An example of aspect **A**: Solve the equation

$|p-x|+|p+x|=2p$, where $p>0$ is a real parameter.

One of the ways of solving this equation is the following: If $\boldsymbol{x}$ is a point of the number line, then the expressions $|p-x|$ and $|p+x|$ actually model the distance from point $\boldsymbol{x}$ to points p and $-p$, respectively. That is why the problem can be reformulated (i.e. transformed into an equivalent problem) in this way: Which are the numbers $\boldsymbol{x}$ on the number line, the sum of the distances of which to the end of the segment [-*p*; *p*] is equal to 2p (the length of this interval)? The answer is clear – each point of the specified interval possesses this property, so the solutions of the equation are the numbers from the interval [-*p*; *p*].

An example for aspect **B**: After the consideration of the following basic problem "Prove that if *P* is a random point on the side *AB* of the triangle *ABC*, then the areas of the triangles *APC, PBC* and *ABC* refer to one another in the same way as do the lengths of the segments *AP, PB* and *AB*", it is expedient to ask the students to formulate a problem for the solution of which this basic problem is used. Here is a problem like this one formulated by students: "If the diagonals of the convex quadrangle *ABCD* are constructed, find the ratio of the areas of the triangles into which the quadrangle is divided by the diagonals."

It becomes clear from the examples that the activities formulating and transforming of problems are indeed a means for the realization of the basic activity (solving of problems), as well as for the formation and perfection of skills for their implementation. But in order to become a means, first they have to be an objective of education. Therefore, these three activities with mathematical problems which are interconnected with respect to both their functions and contents can be considered in a triad in a reflexive-synergetic aspect.

Here by "the **triad** of activities solving, formulating and transforming of mathematical problems" we understand the system of their respective interrelated subsystems of activities. The structure and the separate types of properties (structural, functional and substrative) of this system can be considered dynamically in accordance with their purpose in various pedagogic situations.

While searching for opportunities to apply the ideas of the reflexive-synergetic approach to the "student – teacher" subsystem, various alternatives are put forward, some of which are related to a new interpretation of already familiar methods, forms and means of education. In this connection and with a view to the investigation of the reflexive-synergetic potential of the general logical methods, particular mathematical methods and heuristics as a means of implementing the triad of activities (solving, formulating and transforming of mathematical problems), it has proved expedient to build a conceptual model of this triad. This model is presented schematically in Figure 1. The main components of the model are related to the self-organization and its essential constituent part – the reflexion of the system under consideration. It allows understanding the importance of its bifurcational development, i.e. how its choice of behavior can be influenced with some fluctuations.

In principle, when applying the model, "the channels and jokers method" [7, p. 96] is used in a reflexive-synergetic aspect, according to which the behavior of the whole structure is determined by its projection on a small scale. That actuates the principle of evolutionary holism. The synthesis of simple structures in a complex system in the case considered here (the "student-teacher" system) is developed positively by establishing a general pace of the evolution of the activities solving, formulating and transforming of mathematical problems, i. e. the structures-attractors as future conditions are pre-given, outlined in the present. For that reason, the effect which is achieved by the triad of activities on the basis of the reflexive-synergetic approach is bigger than the summary effect of the three activities when they are realized separately and independently from one another.

The expectations from the research work are that the constructed model of the triad of activities can help to overcome weaknesses in the practical realization of the classical variant – consideration of each and every component (in our case, the three activities with mathematical problems) on its own. And this is accomplished by mutual reflexive-synergetic addition and enrichment of the theory and practice of education.

An idea in this respect is to construct a didactically expedient system of problems designed to introduce students to the general logical methods for solving problems and mastering general and particular methods and heuristics at a respective reflexive level. A reason for this is the fact that the educational process in mathematics at school is organized to a large extent on the basis of a complex of systems of mathematical problems.

That is why the consideration of the issue about the construction of didactic systems of problems with the expedient inclusion of the triad of activities in mathematics education, as well as the introduction of the results of such and other similar studies into practice, can contribute considerably to improving the methodology of work in a theoretical and practical aspect.

In order for the systems and subsystems of problems in question to carry out their purpose, their construction must be predetermined by the structural characteristics of different types of mathematical problems, for example: hierarchical relations between their corresponding hypotheses in their given requirement or among their respective objectives; their solutions must also include the solutions to general sub-problems; the methods for solving problems of a given type must be a part of a more general methodology for solving problems from the next – according to the hierarchy – type (thus better accessibility and continuity of education are secured), etc. In this way, there is ensured a better opportunity for the implementation of logical connections between knowledge-components of systems at a mega level and knowledge- components of systems at a micro level with a view to mastering respective knowledge at a new macro level.

Based on innovative studies of the connections between education and development (specified in [4] particularly for mathematics education) and in virtue of other topical approaches (the reflexive and synergetic ones) developed in [5], [6], [7] and [30], there became possible the investigation of the place of self-organization in the system of cognitive processes related to the mastering of the general logical methods, particular mathematical methods and heuristics for solving mathematical problems. For that purpose, there was developed a method for acquiring and applying mathematical knowledge in the context of the intellectual and praxeological reflection [15], as well as structural models of systems and subsystems of problems with an important didactic function, an operational model for mastering of skills for the application of the basic general logical methods [34], etc. There was constructed a methodological system, too, which contains complex approaches and technological models for a purposeful formation and development of knowledge and generalized skills for the implementation of the prototypical activity (solving problems) on the basis of active inclusion also of elements from the production activities – formulating and transforming of problems, which indisputably plays a crucial role for the intellectual development of secondary school students (respectively, university students on a mathematics education degree program) for the perfection of his/her personality.

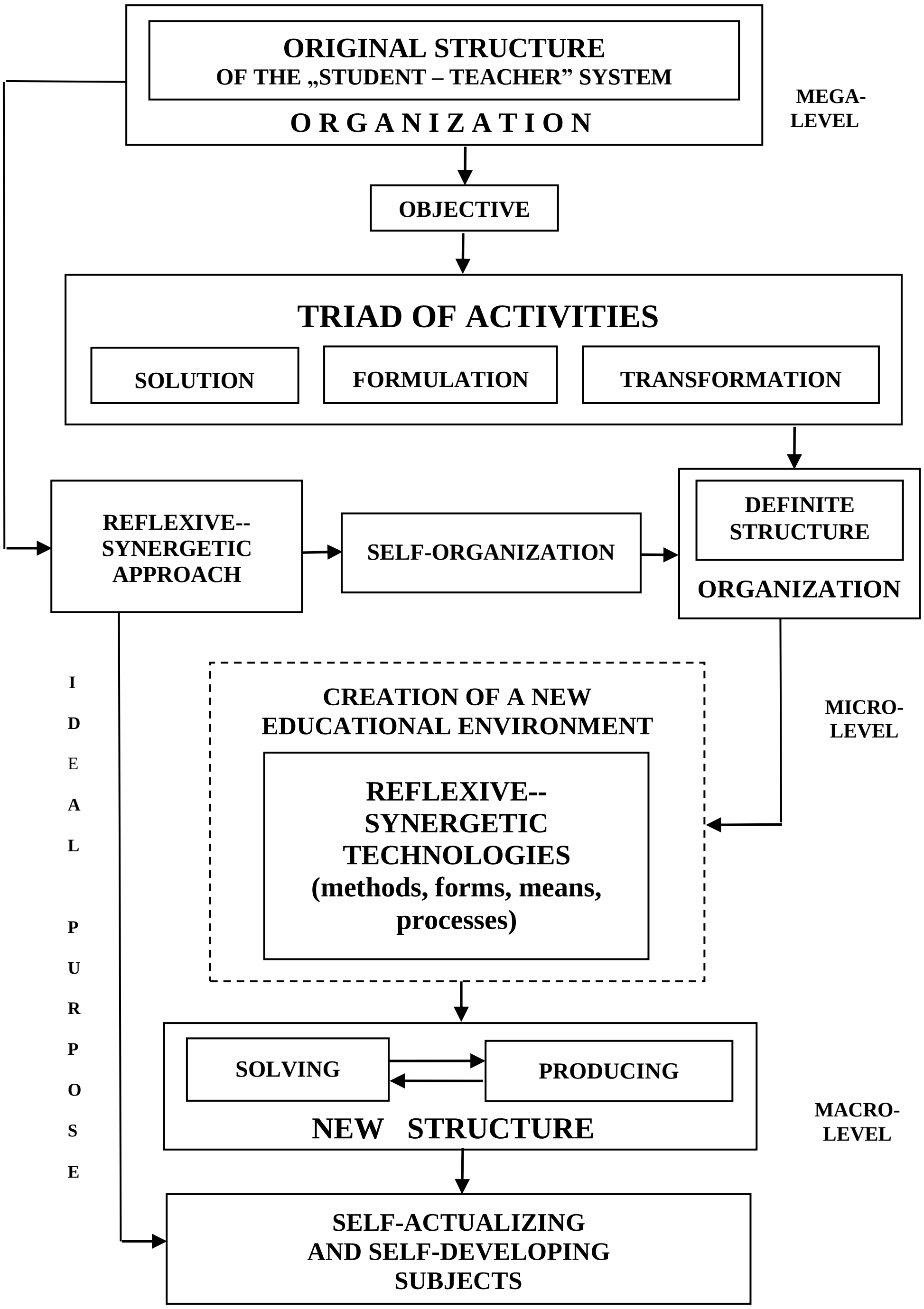

Fig. 1. A conceptual model of the triad solving, formulating and transforming of mathematical problems in the context of the reflexive-synergetic approach

The experimental work shows the positive influence of the combined use of the activities formulating and transforming of mathematical problems on the formation and development in a reflexive-synergetic plan of comprehensive knowledge and skills in secondary-school graduates and undergraduate applicants for performing the basic activity (solving problems), which supports the present thesis: the reflexive-synergetic effect of the combined use of the triad is expressed in independent addition and further building of the separate sub-systems of the system of opportunities, which forms both the student and the teacher as self-organizing people.

It is worth mentioning that the triad is mainly implemented on the basis of a combined use of general logical methods, particular mathematical methods and heuristics, and the basic general logical methods are an important means for a better and faster self-organizing and enhancing the process of self-actualization and self-development in the course of finding and realizing the solutions to problems.

The differentiated triad of the interrelated activities for working with mathematical problems (solving, formulating and transforming) and the constructed conceptual model in the context of the reflexive-synergetic approach (fig. 1) as a basic component of the developed theoretical basis of the methodological system contribute considerably to the utilization of important synergic concepts, ideas and principles in realizing the education and for optimizing the reflexive thinking, self-estimation, self-regulation, and self-perfection of students, and can be used as a methodological basis for future studies.

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