MOTIVATING LIBERAL ARTS STUDENTS TO UNDERSTAND MATHEMATICAL CONCEPTS THROUGH MATHEMATICS QUOTATIONS

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ABSTRACT

Various aspects of using mathematics quotations in mathematics classes to motivate liberal arts students are discussed: historical, philosophical, logical, aesthetic, psychological, pedagogical, as well as interdisciplinary ones.

INTRODUCTION

Providing an effective mathematics instruction to undergraduate students who lean more towards humanities is often challenging for the professors. It depends not only on the abilities of the students to handle the curriculum. Psychological factors as students' personal feelings about mathematics which they carry from their high school period may harden their mathematics education in the university.

The humanistic approach to teaching mathematics includes a proper choice of methods for representing curriculum which are tailored to the audience. Among the instruments that have the potential to express mathematical ideas in a clear, brief and formula-less way are the mathematics quotations. The brightness and imagery of the thoughts make them a special genre that is welcomed by both professional mathematicians and people without mathematical background. Alongside with the nonstandard interpretation of mathematical concepts and their applicability to real life situations, such quotes often show real life situations in the light of mathematical concepts.

MATHEMATICS QUOTATIONS IN TEACHING MATHEMATICS

The wit in mathematics quotations makes mathematical concepts more understandable and impressive for the students. They can be successfully used as a complementary tool to build up the undergraduates' mathematical thinking. Mathematics quotations make the discipline more popular, likeable and grounded for the students and give them the confidence that they can successfully manage with the curriculum. Fleron (1998) reported that he had implemented such quotations in his everyday teaching by starting each of his lectures with a quote of the day written on the chalkboard in advance. He observed that this practice was embraced by his students and they talked about the daily quote even before the beginning of the class.

I also use quotations about mathematics in the mathematics classes I teach to Liberal Arts students [1], but in a different form. It seems more reasonable to me to select quotes, related to a certain topic and to represent them to my audience through the means of Power Point presentations. The topic I have chosen concerns numbers. It allows inclusion of pictures, sound effects and music to emphasize the mathematical concepts and to make them more attractive to the audience. This approach significantly raises the interest of the Liberal Arts students towards the curriculum and increases their motivation for the class.

ASPECTS OF ENRICHING STUDENTS' KNOWLEDGE BY MEANS OF MATHEMATICS QUOTATIONS

My efforts to illustrate as many aspects of mathematical concepts as possible by mathematics quotations made me organize them as follows:

▪ **Historical aspects.** "*The progress of the Society*", underlines Grozdev (2007), "*depends more and more on modern mathematics achievements. The problem here is that the assimilation of mathematical knowledge is a specific process, and one should not disregard History, starting his or her education with modern Mathematics and skipping the classical mathematical results*."

Examples of using mathematical concepts can be found in historical for the humankind events and documents – a fact that helps students realize the power of mathematical literacy. In the Declaration of Independence of the United States (July 4, 1776) Thomas Jefferson (1743-1826), one of its authors, has used Euclid's Elements as a template, although they had been written more than two thousand years before. Addressing the same issue, on April 6, 1859 Abraham Lincoln (1953) made the following comment:

"*The principles of Jefferson are the definitions and axioms of free society*."

For ages the study of mathematics has been regarded as a privilege of free human spirit. John Adams (1797-1801), who is also among the founding fathers of the United States as Washington and Jefferson, has left the generations a remarkable quote:

"*I must study politics and war, that my sons may have liberty to study mathematics and philosophy*."

▪ **Philosophical aspects.** The place of mathematics in the system of human knowledge is broadly discussed among mathematicians and philosophers. In his dissertation T a b o v (2004) notes:

"*Mathematics with its high degree of abstractness as philosophy, for example, could be classified neither under humanities, nor under science, thus making unreasonable to put it closer to science than to humanities. In such sense, when it is talked about a gap in teaching mathematics and humanities, it has to be taken into account that this gap can be overcome, but for the purpose deliberated and practically tested methods are needed*."

Undergraduate students answer the question "What is mathematics?" on the base of their own experience. Thus in my presentation I put some humorous descriptions of mathematics because they regard such important mathematical objects as numbers:

"*If it's green, it's biology. If it stinks, it's chemistry. If it has numbers, it's math. If it doesn't work, it's technology*." (Unknown author)

▪ **Logical aspects.** Probably the most important role of mathematics quotations in a mathematics classroom is to properly illustrate the studied topics. Among the funny quotes I use is the following one:

"*Philosophy is a game with objectives and no rules. Mathematics is a game with rules and no objectives.*" (Unknown author)

It helps my students understand the importance of rules in mathematics. Since mathematical definitions can be interpreted as a kind of rules that specify the properties of mathematical objects, the incorrect dealing with definitions may lead to wrong conclusions. For several years I have encountered the following situation while teaching the topic about prime numbers. When being asked to write several prime numbers, the students usually list 3, 5, 7, 11, 13, … Sometimes they include the numbers 1 and 2 in their sequence. For me this is an opportunity to refresh the audience's mind about what they have studied at school. They recall that number 2 is the only even prime number, but number 1 is not prime by definition. My next step is to ask the students to formulate the definition of a prime number. Here is the answer I get from them most often:

"*Definition 1*". *Numbers from the set* $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ *which are divisible only by* 1 *and themselves are called **prime** numbers*.

To help my audience figure out their mistake on their own, I write their formulation on the chalkboard and ask them to check the status of number 1. The students convince themselves that number 1 satisfies the two conditions of "*Definition 1*", therefore it has to be a prime. This wrong result sheds light on what has been missed by the audience and they suggest the following correction:

"*Definition 2*". *Numbers from the set* $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ *which have exactly two different divisors are called **prime** numbers*.

In this new situation I recommend my students to use number 1 as a test example again. It has two different integer divisors: 1 and (−1). Thus it follows once more that number 1 is a prime number. Finally the students guess why their "*Definitions*" are defective and give even two correct formulations:

Definition 1. *Numbers from the set* $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ *which are greater than 1 and are divisible only by* 1 *and themselves are called* ***prime*** *numbers*.

Definition 2. *Numbers from the set* $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ *which have exactly two different positive divisors are called* ***prime*** *numbers*.

▪ **Aesthetic aspects.** The ancient philosopher Proclus (412-485) has said:

"*Wherever there is number, there is beauty*."

After the definition of prime numbers has been clarified, I introduce my audience to one of the most striking results in number theory – the theorem about the infinity of primes. Its proof allows various approaches, which Aigner and Ziegler (2003) show in the very beginning of "The Book". Euclid's proof impressed my students with its brilliant simplicity and they made a comment that six graders could also understand it.

Ancient Greek mathematicians have been attracted by one special class of numbers, equal to the sum of all their positive divisors, which are less than the numbers themselves. To name those, they have used even a higher aesthetical category than *beauty* and called them *perfect*:

$$6 = 1+2+3,$$
$$28 = 1+2+4+7+14,$$
$$496 = 1+2+4+8+16+31+62+124+248,$$
$$8128 = 1+2+4+8+16+32+64+127+254+508+1016+2032+4064, \text{ etc.}$$

The next perfect number is 33 550 336, which shows that in the interval $(10^5, 10^7)$ there is no representative of that "breed". This fact has made the French mathematician René Descartes (1596-1650) to exclaim:

"*Perfect numbers like perfect men are very rare*."

Since I used this quotation in a mathematics class and Descartes did not specify what he had meant under the notion of *perfect men,* I was forced to clarify it through another saying made three centuries after him:

"*Perfection is what American women expect to find in their husbands... but English women only hope to find in their butlers*." (W. Somerset Maugham)

Such a humorous play with the notions in mathematical and everyday life sense initiated an interesting discussion on how my students understand the beauty of mathematics. Probably suffered a lot from boring mathematical activities at school, the students pointed out as a main feature of mathematical beauty the *presence of a bright idea, which unexpectedly* "*cuts the knot*" *in reasoning and crucially shortens the solution of the problem*.

▪ **Deepen the subject knowledge.** Mathematics quotations by world-known scholars or proponents of science carry condensed expert knowledge and often give unusual perspectives to mathematical ideas. An important property of the infinite set $\mathbb{N}$ of all natural numbers (as well as of the set $\mathbb{Q}$ of all rational numbers) is its *countability*, in contrast with the *uncountable* set $\mathbb{R}$ of all real numbers. Via the set of all subsets of $\mathbb{N}$, Georg Cantor (1845-1918) created a connection between these completely different types of infinity: if cardinality of $\mathbb{N}$ is denoted by $\aleph_0$ and cardinality of $\mathbb{R}$ by $\aleph_1$, then it can be written that $\aleph_0 < \aleph_1$. Thus in the realm of ***transfinite numbers*** exists a similar ordering $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$ with no transfinite "gaps" between, as in the set of ***natural numbers*** $1 < 2 < 3 < \ldots$ with no free "integer places" between.

The amazing idea of Cantor excited my audience and during the upcoming classes I was asked to discuss it again. Some of the students even brought friends that had not been enrolled in the class to listen about the hierarchy of infinities. Their interest inspired me to quote the notorious Austrian physicist and philosopher Ernst Mach (1838-1916), who was a contemporary of Cantor:

"*Mathematics may be defined as the economy of counting. There is no problem in the whole of mathematics which cannot be solved by direct counting*."

In my lecture about numbers the students were impressed by the different properties that the two classes of real numbers, ***rational*** and ***irrational***, possess. After my audience learned that on their part the irrational numbers are also divided into two classes: ***algebraic*** and ***transcendental***, they vividly admired the metaphor by the journalist Richard Preston (1992):

"*There is no finite algebraic equation built from whole numbers that will give an exact value for pi. If equations are trains threading the landscape of* (these) *numbers, then no train stops at pi*."

▪ **Psychological and pedagogical aspects.** Probably in any mathematics class there are students who do not fully understand what is being taught. Although homeschooled by her father, the Dame of the criminal genre Agatha Christie (1890-1976) has felt so bored with her everyday mathematical exercises, that in her Autobiography she has written:

"*I continued to do arithmetic with my father, passing proudly through fractions to decimals. I eventually arrived at the point where so many cows ate so much grass, and tanks filled with water in so many hours I found it quite enthralling*."

The reflections of the undergraduate students on mathematics classes can also indicate whether there are difficulties with the curriculum, disappointment by the teaching methods or mathematical anxiety. They show the professors how the learning outcomes can be improved. Regarding the way mathematical content has to be taught to the students, the American professor Stan Gudder recommends:

"*The essence of mathematics is not to make simple things complicated, but to make complicated things simple*."

From more artistic point of view, the poet Dorothy Parker suggests:

"*The cure for boredom is curiosity. There is no cure for curiosity*."

In a broader sense, "*the cure*" can be an interesting topic, group work, inquiry based activities, unexpected methods of teaching, etc.

There are not a lot of mathematical textbooks in whose preface the students read an instruction like this:

"*If you fail* (to understand a paragraph), *even after three readings, very likely your brain is getting a little tired. In that case, put the book away, and take to other occupations, and next day, when you come to it fresh, you will very likely find that it is quite easy*."

It has been given by the mathematician Lewis Carroll (1832-1898), the author of "Alice's adventures in Wonderland" and "Through the looking glass", to the readers of his book "Symbolic Logic".

▪ **The gender issue.** In my classes neither the male, nor the female students have encountered problems to ask questions, to express opinions, to make an appointment for some extra work or to achieve good results. The examples of women with extraordinary career in mathematics I give in class aim to show that mathematical achievements do not have gender. Therefore, my purpose of using quotations about women's behaviour in mathematical environment is to improve the atmosphere in the classroom. Besides that, I want to pass on to the students the idea that we talk about women in mathematics solely because there are men there as well. Here is the opinion of the prominent topologist Mary Ellen Rudin, who is also an excellent teacher and a mother of four, why female mathematicians in academics still remain underrepresented (Albers et al., 1990):

"*Mathematics is obviously something that women should be able to do very well. It's very intuitive. You don't need a lot of machinery, and you don't need a lot of physical strength. You just need stamina, and women often have a great deal of stamina. So why do not more women become mathematicians? I think that for some reason, probably sociological, girls are refusing to look – they simply won't try something that they view as a hard problem in mathematics. But boys for some reason are willing and eager to look at the hard problems*."

The French journalist and playwright Marcel Achard (1899-1974) also had no doubt that women possess inborn knack for mathematics. To his full of friendly humour quotation I allowed myself to add a phrase, just to have the four arithmetic operations available:

"*Women have a passion for mathematics. They divide their age in half, double the price of their clothes*", subtract something from their weight "*and always add at least five years to the age of their best friend*."

▪ **Interdisciplinary aspects: mathematics and art.** The process of motivating Liberal Arts students to study mathematics is more effective when we talk about the relations between mathematics and other disciplines. For example, architects as M. C. Escher, Buckminster Fuller, Ernő Rubik owe their freedom and creativity in the three dimensional space to great extent to their understanding of mathematical concepts. Vice versa, their works are also a source of inspiration for further development of mathematics.

A contemporary of Lobachevsky (1792-1856), the outstanding poet of Russia Alexander Pushkin (1799-1837) was deeply excited by the spirit of the epoch when non-Euclidean geometry was born. Here is what Pushkin has written:

"*Inspiration is needed in geometry, just as much as in poetry*."

The ancient geometer Pythagoras is well known with his musical scale. In a way, accessible for students, Kelevedjiev and Dzhenkova (2007) explain the mathematical concepts behind the structure of Pythagoras' scale and the well-tempered scale. The authors make musical analogies of such popular mathematical notions and objects, as *Golden Mean* and *Fibonacci numbers* and provide the readers with a lot of examples.

Addressing the relationship between mathematics and music, Gottfried Leibniz (1646-1716) notes:

"*Music is the pleasure the human soul experiences from counting without being aware that it is counting*."

Mason Cooley (1927-2002), a professor in English language, speech and literature, has created a wonderful parallelism:

"*Mathematics: silent harmonies. Music: sounding numbers.*"

Such quotations may decisively alter the "gastronomical" idea of fractions as pizza slices being taught at school, to a more spiritual level.

A funny way to show my students that mathematics is not just a discipline to study, but also a way to think is the example of one special kind of primes, introduced by the following:

Definition. *Two prime numbers form a* **sexy pair**, *if the absolute value of their difference is equal to 6.*

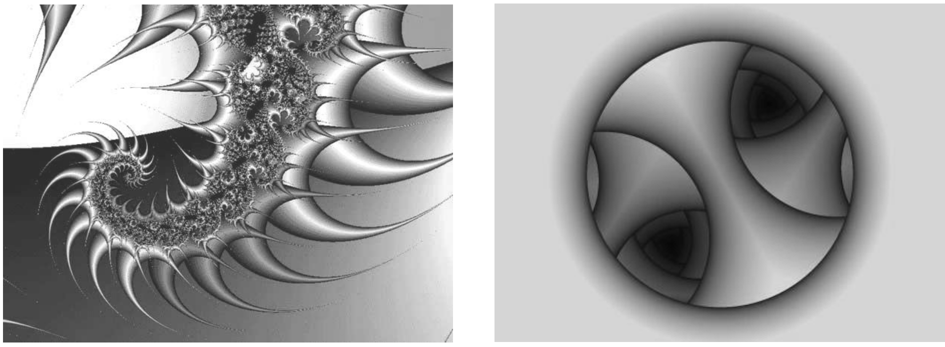
Following the definition, the students find several *sexy* pairs: *5* and *11*, *7* and *13*, *17* and *11*, *17* and *23*, *31* and *37*, etc. After that they start wondering why the pairs had been called ***sexy***. The answer is that this mathematical definition is based on a linguistic situation: *sex* is the Latin word for *six*. Not to leave the linguistic idea isolated, I tell my audience the joke: "*The Romans did not find algebra very challenging, because X for them was always 10.*"

Discussing the connections between mathematics and linguistics, I share with my class the curious story about Josiah Willard Gibbs (1839-1903), one of the greatest American scientists. Focused on his research, he rarely gave speeches. But sitting in a committee whose aim was to improve the American education by cutting off hours from mathematics and giving them to the study of foreign languages, he could not bear to be silent. Here is Gibbs' historical speech:

"*Mathematics* ***is*** *a language*."

The topic on numbers gives me also an opportunity to say something about the language *of* mathematics. As dimensions of special sets, named *fractals*, the nonnegative rational numbers open up a whole new field of application: fractal geometry. When I asked my students whether they had heard about *fractals*, one of them answered: "*Yeah, those kinds of lizards, you know.*" I agreed and showed such "a creature" to the class (***Fig. 1-a***). But when I showed the next picture [2] (**Fig. *1-b***), the immediate reaction of the same student was: "*This is Art!*" In such situation, hardly anything more appropriate can be said except the words of Benoit Mandelbrot (1977), the founder of fractal geometry:

"*Being a language, mathematics may be used not only to inform but also, among other things, to seduce*…"



***a)* "*Those kinds of lizards…*"** ***b)* "*This is Art!*"**

Fig. 1.* A student's perception of *fractals

ON THE AFTERMATHS OF USING MATHEMATICS QUOTATIONS IN MY MATH CLASSES

"*Education is what survives when what has been learned has been forgotten*", has said the psychologist Burrhus F. Skinner (1904-1990). The recognition I receive from the audience immediately after my presentation is not only through their applauses. Many of the students tell me how much they appreciate the time and efforts I had invested to research for appropriate quotes and put them together. Two months after my presentation on numbers, one of the students showed his peers a similar in style presentation about mathematics and music. In a year, already a former student of mine sent me via e-mail her Power Point presentation on Business. There she had used mathematical concepts and quotations to structure her ideas and attract the attention of her audience. Thus the class showed in their own way to me that they had understood my message through the quotation:

"*Mathematics is not a spectator sport*." (Unknown author)

To these students mathematics did become a language to communicate. They used the sources I recommended in class, mostly the websites of Furman University [3] and Westfield State College [4], the books by Dimovski (1972), Gaither and Cavazos-Gaither (1998), etc.

QUOTATION ABOUT QUOTATIONS: CONCLUDING REMARKS

Wherever one digs in Lewis Carroll's books, there is a quote. The following dialogue from the novel "Sylvie and Bruno" (Carroll, 2002) is a short mathematical story about the value and lifespan of quotations:

"*Which contain the greatest amount of Science, do you think, the books, or the minds*?"

"*If you mean **living** minds, I don't think it's possible to decide. There is so much **written** Science that no living person has ever **read**: and there is so much **thought-out** Science that hasn't yet been **written**. But, if you mean the whole human race, then I think the minds have it: everything, recorded in books, must have once been in some mind, you know*."

"*Isn't that rather like one of the Rules in Algebra*? *I mean, if we consider thoughts as **factors**, may we not say that the Least Common Multiple of all the minds contains that of all the books; but not the other way*?"

"*Certainly we may if we could only **apply** that Rule to books*! *You know, in finding the Least Common Multiple, we strike out a quantity wherever it occurs, except in the term where it is raised to its highest power. So we should have to erase every recorded thought, except in the sentence where it is expressed with the **greatest intensity***."

It is exactly *the intensity of thoughts* I seek to boost in my Liberal Arts students through mathematics quotations. My intention is to show them this treasury of concepts and beauty which can broaden their horizons of thinking. I do hope that in the future they will search for the numerous new websites with mathematics quotations that constantly appear on the Internet. This trend on the Web is evidence that the modern world needs the gems of the *beautiful minds* that have been preserved for us throughout the ages.

NOTES

[1] The course named "*Classical and modern ideas in mathematics*" I teach in cooperation with Prof. D. Sc. Jordan Tabov at New Bulgarian University in Sofia.

[2] http://www.root.cz/clanky/obsah-jednotlivych-casti-serialu-a-galerie-fraktalu-ii/ (accessed in January, 2009).

[3] http://math.furman.edu/~mwoodard/mqs/mquot.shtml (accessed in January, 2009)

[4] http://www.wsc.mass.edu/math/faculty/fleron/quotes (accessed in January, 2009).

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