# GENERALIZED NON-ARCHIMEDEAN TWIN CIRCLES OF THE ARBELOS

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## ABSTRACT

*We generalize two pairs of non-Archimedean twin circles of the arbelos.*

## 1. INTRODUCTION

Let O be a point on the segment AB in the plane and $\alpha$, $\beta$ and $\gamma$ the semicircles on the same side of the diameters AO, BO and AB, respectively. The area surrounded by the three semicircles is called an arbelos or a shoemakers' knife. Let I be the intersection of the perpendicular of AB through O and $\gamma$. The line OI divides the arbelos into two areas and the two inscribed circles are congruent (see Figure 1). The circles are called twin circles of Archimedes and circles congruent to the twin circles are called Archimedean circles.

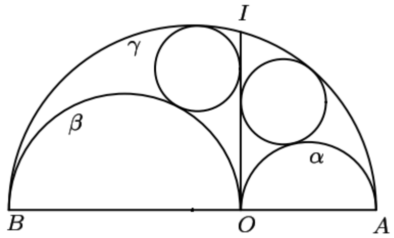


Figure 1.

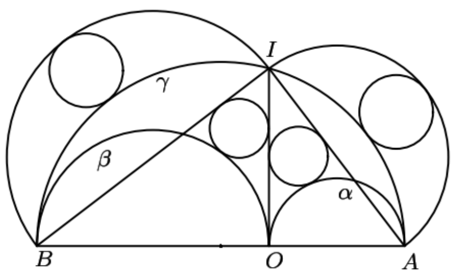


Figure 2.

Let a and b be the respective radii of the circles $\alpha$ and $\beta$. The radii of Archimedean circles are ab/(a+b). In [1] we have shown the following two facts (see Figure 2):

(a) The incircle of the curvilinear triangle made by OI, IA and $\alpha$ is congruent to the incircle of the curvilinear triangle made by OI, IB and $\beta$ with common radii

$$\left(\sqrt{a}+\sqrt{b}-\sqrt{a+b}\right)^2.$$

(b) The maximal incircle of the lune made by $\gamma$ and the circle with a diameter IA is congruent to the maximal incircle of the lune made by $\gamma$ and the circle with a diameter IB with common radii

$$\frac{1}{2}\sqrt{a+b}\left(\sqrt{a}+\sqrt{b}-\sqrt{a+b}\right).$$

Therefore, these are twin circles, though not Archimedean. In this paper we generalized both facts.

## 2. A GENERALIZATION OF (a)

In this section we generalize (a) (see Figure 3).

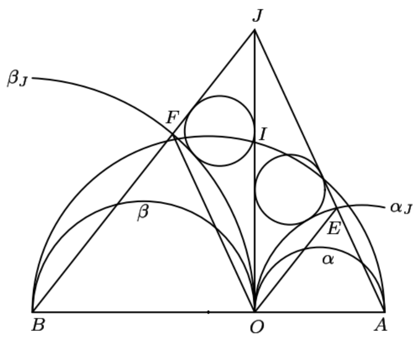


Figure 3.

**Theorem 1.** *For a point J on the line OI (J ≠O), E and F are points on the line AJ and BJ respectively such that OEJF is a parallelogram.* $\alpha_J$ *and* $\beta_J$ *are circles touching OJ at O passing through E and F respectively. If* $|OJ|=k|IO|$ *for a real number k, then the incircles of the curvilinear triangle formed by* $\alpha_J$*, AJ, OJ is congruent to the incircle of the curvilinear triangle formed by* $\beta_J$*, BJ, OJ with common radii*

$$\frac{\left(\sqrt{ab}-\left(\sqrt{k^2a+b}-k\sqrt{a}\right)\left(\sqrt{a+k^2b}-k\sqrt{b}\right)\right)^2}{a+b}.$$

Proof. We set up a rectangular coordinate system with origin O, such that A and I lie on the positive parts of x-axis and y-axis, respectively. Let (x,y) be the coordinates of the points E. From the two similar triangles JBA and EOA, we get $2(a+b) : 2a = 2k\sqrt{ab} : y$ and $x : y = 2b : 2k\sqrt{ab}$. Therefore, the coordinates of the points E are $(2ab/(a+b), 2ka\sqrt{ab}/(a+b))$. Let r be the radius of the circle $\alpha_J$. Since $\alpha_J$

passes through E, we get

$$r = \frac{a(k^2a+b)}{a+b}. \qquad (1)$$

Let $s$ and $P$ be the radius and the center of the incircle of the curvilinear triangle made by $\alpha_J$, $AJ$, $OJ$. The coordinates of $P$ is $(s, 2\sqrt{rs})$. Since $\overrightarrow{JO} = \left(0, -2k\sqrt{ab}\right)$, $\overrightarrow{JP} = \left(s, 2\sqrt{rs} - 2k\sqrt{ab}\right)$, $\overrightarrow{JA} = \left(2a, -2k\sqrt{ab}\right)$, and $\overrightarrow{JP}$ bisects the angle between $\overrightarrow{JO}$ and $\overrightarrow{JA}$, we get

$$\frac{-2k\sqrt{ab}\left(2\sqrt{rs}-2k\sqrt{ab}\right)}{2k\sqrt{ab}} = \frac{2as - 2k\sqrt{ab}\left(2\sqrt{rs}-2k\sqrt{ab}\right)}{\sqrt{4a^2+4k^2ab}}.$$

Substituting (1) and rearranging we get

$$s - \frac{2\sqrt{k^2a+b}\left(k\sqrt{b}-\sqrt{a+k^2b}\right)}{\sqrt{a+b}}\sqrt{s} + 2k\sqrt{b}\left(k\sqrt{b}-\sqrt{a+k^2b}\right) = 0.$$

The left side can be factored:

$$\left(\sqrt{s} - \frac{\sqrt{ab} - \left(\sqrt{k^2a+b} - k\sqrt{a}\right)\left(\sqrt{a+k^2b} - k\sqrt{b}\right)}{\sqrt{a+b}}\right)$$

$$\times\left(\sqrt{s} - \frac{-\sqrt{ab} - \left(\sqrt{k^2a+b} + k\sqrt{a}\right)\left(\sqrt{a+k^2b} - k\sqrt{b}\right)}{\sqrt{a+b}}\right) = 0.$$

Since $-\sqrt{ab} - \left(\sqrt{k^2a+b} + k\sqrt{a}\right)\left(\sqrt{a+k^2b} - k\sqrt{b}\right) < 0$, we get

$$\sqrt{s} = \frac{\sqrt{ab} - \left(\sqrt{k^2a+b} - k\sqrt{a}\right)\left(\sqrt{a+k^2b} - k\sqrt{b}\right)}{\sqrt{a+b}}.$$

□

The x-coordinate of E in the proof suggests that we can construct an Archimedean circle passing E and touching OJ. We hope to discuss this in a later paper. The condition J being on the line OI is not essential (see Figure 4).

**Theorem 2.** *For a point J, which does not lie on the line AB, E and F are points on the line AJ and BJ respectively such that OEJF is a parallelogram. If* $\alpha_J$ *and* $\beta_J$ *are circles touching OJ at O passing through E and F respectively, then the incircle of the curvilinear triangle formed by* $\alpha_J$*, AJ, OJ is congruent to the incircle of the curvilinear triangle formed by* $\beta_J$*, BJ, OJ.*

Proof. Let A' and B' be the respective intersections of the line perpendicular to OJ passing through O and the lines OA and OB. If we consider the arbelos formed by the circles with diameters OA', OB' and A'B', then the theorem follows from Theorem 1.

□

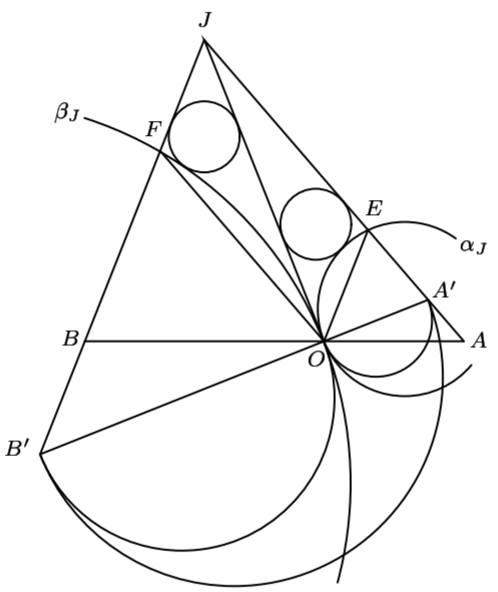


Figure 4.

## 3. A GENERALIZATION OF (b)

For two intersecting circles $\alpha$ and $\beta$, $\alpha\beta$ is the maximal circles contained in $\alpha$ and $\beta$, and $\overline{\alpha}\beta$ is the maximal circle which is not contained in $\alpha$ but $\beta$ (see Figure 5). Our generalization of (b) is the following (see Figure 6).

**Theorem 3.** *For a triangle $ABC$, let $\alpha$, $\beta$ and $\gamma$ be circles with diameters $BC$, $CA$ and $AB$ respectively. Then $\gamma\alpha$, $\alpha\overline{\beta}$ and $\overline{\beta}\gamma$ are congruent. Similarly each of $\alpha\beta$, $\beta\overline{\gamma}$, $\overline{\gamma}\alpha$ and $\beta\gamma$, $\gamma\overline{\alpha}$, $\overline{\alpha}\beta$ is a congruent triplet.*

*Proof.* Let $E$ and $F$ be the midpoints of $BC$ and $CA$ respectively. The radius of $\alpha\overline{\beta}$ is equal to

$$\frac{1}{2}(|FE| + \frac{1}{2}|BC| - \frac{1}{2}|CA|) = \frac{1}{4}(|AB| + |BC| - |CA|). \tag{2}$$

Similarly the radii of $\overline{\beta}\gamma$ and $\gamma\alpha$ are expressed by (2). □

If A, B and C are collinear, then the three congruent circles of each of the triplets are point circles A, B and C, or they coincide.

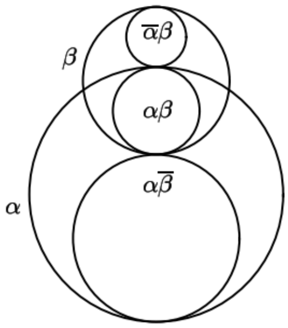


Figure 5.

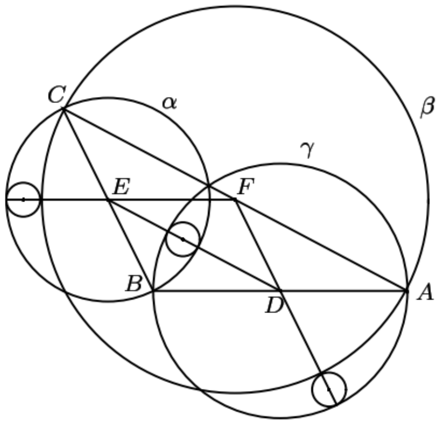


Figure 6.

## 4. CONCLUSION

The arbelos has been offering many comprehensible topics, which can be understood by high school students. Since most of those topics are developing today, the students can see the actual development of mathematics in real time. From an educational point of view, it seems to give a good motivation for studying mathematics.

## REFERENCE

[1] Hiroshi Okumura and Masayuki Watanabe, Non-Archimedean twin circles of the arbelos, Mathematics Plus Journal **16** (63) pp.63-64 (2008).