# PERELMAN'S GEOMETRIC METHOD OF SOLVING LIQUID POURING PROBLEMS

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## ABSTRACT

*A method of solving liquid poring problems with three vessels (Poisson's problem) of the kind in which the biggest vessel is full, while the other two with volumes a and b are empty and $c \geq a + b$ is presented in the paper. The method is based on elementary geometric constructions used by J. I. Perelman and to an extent represents "a strict order" (algorithm) of solving. This method is designed for and tested in the classes of popular math with students in the Primary Education degree programs of University of Sofia "St. Kliment Ohridski" and "St Cyril and St. Methodius" University of Veliko Tarnovo.*

## BIOGRAPHICAL DATA OF JAKOV ISIDOROVICH PERELMAN

J. Perelman was born on December, the $4^{th}$, 1882 in Belostok, Republic of Belarus. His father worked as an accountant in a linen factory and his mother was a primary teacher. Hard-working and taught by a highly able pedagogic team, he formed knowledge and skills of independent thinking and scientific research as early as in his student years.

Perelman's activity of popularizing science began in his student years. In 1899 he wrote his first popular science work, driven by rumors of "fire rain" which would end the world. In his text he explained the pending phenomenon and compromised the "profit's predictions". In the form of improvised discussion, combining reliable calculations and easy and understandable comparisons, he told the readers about the Leonidi meteor stream. The annual character of the event was also explained, as well as its harmlessness to humans.

Although he graduated with distinction from the Forest Institute in Saint Petersburg and gained the title "1st-class Forest Scientist", Perelman never worked as a forester. He became a journalist and as soon as 1904 was appointed executive secretary of the journal "Human and Nature". There he published his works and also works and discoveries of other famous scientists. His early materials were in the field of astronomy. Gradually, he extended his interests and publications on math, physics and technology began appearing.

In 1913 the first tome of Perelman's book "Popular Physics" was published. It had a stunning success among readers and physicists. The Petersburg University physics professor Orest D. Hvolson encouraged the young author to continue his work. His popularizing abilities impressed even the inventor of the Russian rocket engine V. P. Glushko. He described him as "a singer of mathematics, musician of physics and poet of astronomy. Perelman developed his own methodology that allowed the reader to easily access scientific facts from various fields of study.

Perelman's work consists of over 1000 articles and essays published in different issues. Besides, he has 47 popular, 40 scientific and 18 text books. After "Popular physics" came "Popular arithmetic", "Popular algebra", "Popular astronomy", "Popular geometry" and "Popular mechanics". His "Popular physics" was reissued almost 30 times in Russian alone.

A considerably important moment of Perelman's development as a science promoter was the opening of St. Petersburg's House of popular science in 1935. This temple of scientific amusement became the favorite spot not only for Perelman, but also for St. Petersburg's students. There they could discover many achievements in science and technology in an accessible and entertaining way.

In 1942 J. I. Perelman and his wife died during the Second World War siege of St. Petersburg.

Perelman did not make scientific discoveries, nor does he have any technical inventions. He did not have any titles or ranks. However, he was true to science and for 43 years he brought satisfaction to all knowledge-thirsty and curious people. His books remain and nowadays are still read with interest. Apart from Russian, they are printed in German, English, Spanish, Portuguese, Italian, Czech, Bulgarian, Finnish and other languages.

## POISSON'S PROBLEM

One of the problems in S. Koval's book "Knowledge through entertainment" is the one of the French mathematician Poisson. What makes an impression is that the

author presents the text and then its synthetic solution without any specific explanation or clues for the reader leading to the problem's solving.
Poisson's problem. During an excursion, one of the tourists bought a can of wine of 8 quarts. The wine quantity had to be divided into halves. How did that happen, given that there were only two vessels in the pub – one of two quarts, and an other of 3 quarts? How many times had the wine to be poured from one vessel to another?

Solution. To divide the quantity in halves, one has to pour 7 times. The vessels are assigned to A – 8 quarts, 5 – 5 quarts, C – 3 quarts. The pours are the following:

| A | B | C |
|---|---|---|
| 6 | 0 | 2 |
| 3 | 5 | 0 |
| 1 | 5 | 2 |
| 3 | 2 | 3 |
| 1 | 4 | 3 |
| 6 | 2 | 0 |
| 4 | 4 | 0." [1] |

There are similar problems in the contemporary math textbooks and workbooks, basically in the 2nd to 7th grades. Unfortunately, only synthetic solutions are again displayed. They lack specific methodical guides and a common approach to solving this kind of problem. For example, in the chapter named "Problems of popular and logical nature" in [3] some liquid pouring is included. Their solutions consist of the order and number of pours only.
The word problems, commonly describing liquid pouring situations are various. Thus, the issue of their classification occurs. While some of them have more than one solution, others do not have a solution. There also stands the problem of the order of pours in all cases no matter the volume of the used vessels.
We suggest the division of the problems with three vessels with volumes $a$, $b$ and $c$ into two kinds: 1. Problems, describing a situation in which the biggest vessel with volume $c$ is initially full and the other two are empty $c \geq a + b$. 2. Problems, describing a situation in which all the three vessels contain an amount of liquid.
The purpose of the current article is to present a method of solving the first kind of problems with the aid of elementary geometric construction. The latter is based on the little known idea of J. Perelman and its essence is an order, algorithm of the vessel to vessel pours.
The core of Perelman's method is the drawing of a rhombic mesh with an acute angle of 60º, which a parallelogram or a pentagram with two parallel sides is applied on. In Perelman's model in these shapes a billiard ball moves which hits the walls and rebounds from them according to the equal angles principle. This motion is presented by rays. Thus, a composition of rays is formed inside the figure.
The problem that Perelman used to illustrate his method was the following:

"Divide the contents of a 12 gallon barrel into halves, using two empty barrels with volumes 9 and 5 gallons" [4]
The author suggest: "Let us draw a figure with side OA containing 9 boxes, OB – 5 boxes, AD – 3 boxes (12 – 9 = 3), BC – 7 boxes (12 – 5 = 7) … Note that each point of the sides of the figure stays at a definite number of boxes away from the sides OA and OB. … Let us suppose that the first point, i.e. the distance between it and OB in number of boxes equal the gallons of water in the 9-gallon barrel, the second, i.e. the number of boxes between the point and OB equals the gallons of water in the 5-gallon barrel. The remaining quantity of water will obviously be contained in the 12-gallon barrel … The first point of impact during the "billiard ball's movement" is A(9, 0), therefore the first pour would distribute the water in the following way:

| | |
|---|---|
| 9 - gallon | 9 |
| 5 - gallon | 0 |
| 12 - gallon | 3 |

Table 1

This is practically possible". [4]
In the purpose of greater clarity, we suggest substitution of Perelman's billiard ball with a ray of light which will be named "pouring ray"
In this way, the "pouring ray" continues its motion along the diagonal of a rhomb from point A to C (4, 5). Therefore, the result after the second pour is:

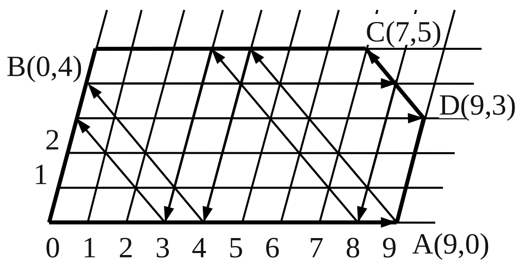


Fig. 1

| | | |
|---|---|---|
| 9 – gallon | 9 | 4 |
| 5 – gallon | 0 | 5 |
| 12 – gallon | 3 | 3 |

Table 2

This is also possible to fulfill.
The third point of ray reflection is (4, 0). So the third pouring should return 5 gallons in the 12-gallon barrel.

| | | | |
|---|---|---|---|
| 9 – gallon | 9 | 4 | 4 |
| 5 – gallon | 0 | 5 | 0 |

| | | | |
|---|---|---|---|
| 12 – gallon | 3 | 3 | 8 |

Table 3

The fourth point B(0,4) informs about the fourth pour.

| | | | | |
|---|---|---|---|---|
| 9 - gallon | 9 | 4 | 4 | 0 |
| 5 - gallon | 0 | 5 | 0 | 4 |
| 12 - gallon | 3 | 3 | 8 | 8 |

Table 4

If this way of operation continues, we come to the next table

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9-gallon | 9 | 4 | 4 | 0 | 8 | 8 | 3 | 3 | 0 | 9 | 7 | 7 | 2 | 2 | 0 | 9 | 6 | 6 |
| 5-gallon | 0 | 5 | 0 | 4 | 4 | 0 | 5 | 0 | 3 | 3 | 5 | 0 | 5 | 0 | 2 | 2 | 5 | 0 |
| 12-gallon | 3 | 3 | 8 | 8 | 0 | 4 | 4 | 9 | 9 | 0 | 0 | 5 | 5 | 10 | 10 | 1 | 1 | 6 |

Table 5

Therefore, the problem is solved: after 18 pours in both the 9-gallon and 12-gallon barrels there are 6 gallons of water.

The same problem can be solved if the “pouring ray” first moves along side OB of the figure and then along the rhomb diagonals till it reaches a point on side AB. In this case less side reflections are made. Again the condition of the equal forth and back angles is met. In this case the problem is solved with less pouring – nine times.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 - gallon | 0 | 7 | 7 | 2 | 2 | 0 | 9 | 6 | 6 |
| 5 - gallon | 5 | 5 | 0 | 5 | 0 | 2 | 2 | 5 | 0 |
| 12 - gallon | 7 | 0 | 5 | 5 | 10 | 10 | 1 | 1 | 6 |

Table 6

The geometrical model shows that the same problem can be solved in two ways, with possible difference between the count of stages in the solution. Moreover, given the solution tables of each specific problem, new problems can be configured, for instance: “Out of three vessels with relevant volumes – 9, 5 and 12 gallons, given that the big one is full of water, one has to extract 1 or 2 gallons with the minimal possible number of pours.”

The geometrical solution model of the problems of kind $c \geq a + b$, where $c$ has the largest volume and is full at the beginning, while $a$ and $b$ are empty, can be enriched by the drawing of an additional figure which depicts the count of lines, representing the remaining quantity in the biggest vessel after each pour. These ideas are illustrated by the following problems:

**Example 1.** There are three vessels – an 8 gallon, a 5 gallon and a 3 gallon one. The first is full while the rest are empty. What is the minimal number of pours to divide the quantity in two equal portions using only the given vessels? [3]

The geometrical solution of this problem will be presented by a drawing which shows the motion of the "pouring ray" on the corresponding side OA or OB of shape OACB with initial point O, i.e. a coordinate system with base point O(0,0). The geometrical constructions and pouring representations are made similar to the previous problem. However, in order to geometrically interpret the quantity in the biggest vessel, next to the basic figure OACB, another one - BCM is drawn as an application. On ray OB, the length of intersecting lines between B and M is equal to the distance from O to A on ray OA. A diagonal line is then drawn through M to intersect AC. The count of intersects dividing the reflection point of each pouring ray from line CM of the additional figure (a triangle in this case) equals the liquid quantity in the biggest vessel.

**Case I.** The pouring ray begins moving along axis OA

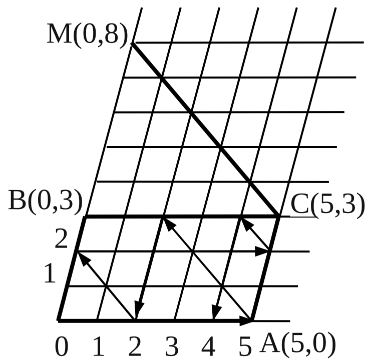


Fig. 2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5-gal. vessel | 0 | 5 | 2 | 2 | 0 | 5 | 4 | 4 |
| 3-gal. vessel | 0 | 0 | 3 | 0 | 2 | 2 | 3 | 0 |
| 8-gal. vessel | 8 | 3 | 3 | 6 | 6 | 1 | 1 | 4 |
| pour | start | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th |

Table 7

**Case II.** The pouring ray begins moving along axis OB

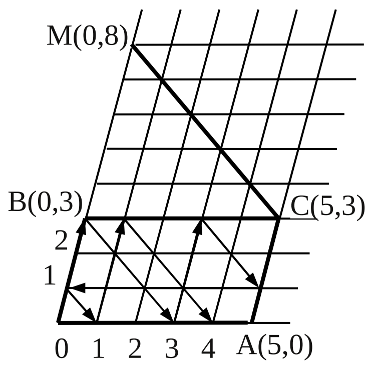

Fig. 3

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5-gal. vessel | 0 | 0 | 3 | 3 | 5 | 0 | 1 | 1 | 4 |
| 3-gal. vessel | 0 | 3 | 0 | 3 | 1 | 1 | 0 | 3 | 0 |
| 8-gal. vessel | 8 | 5 | 5 | 2 | 2 | 7 | 7 | 4 | 4 |
| pour | start | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th |

Table 8

The problem's answer is 7 pours (the case I solution).

**Example 2.** There are 3 vessels with volumes 2 liters, 3 liters and 8 liters. The biggest one is full of punch, the rest two are empty. What is the smallest count of pours needed to get 4 liters of punch, using the three vessels only? [3]

**Case I.** The "pouring ray" initiates its move on the OA axis.

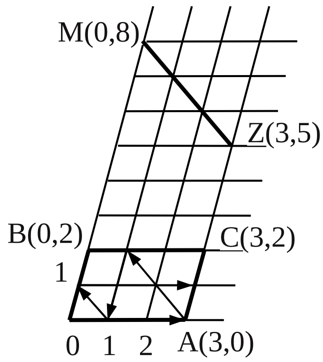


Fig. 4

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 l. vessel | 0 | 3 | 1 | 1 | 0 | 3 |
| 2 l. vessel | 0 | 0 | 2 | 0 | 1 | 1 |
| 8 l. vessel | 8 | 5 | 5 | 7 | 7 | 4 |
| pour | beginning | 1st | 2nd | 3rd | 4th | 5th |

Table 9

**Case II.** The "pouring ray" starts along axis OB

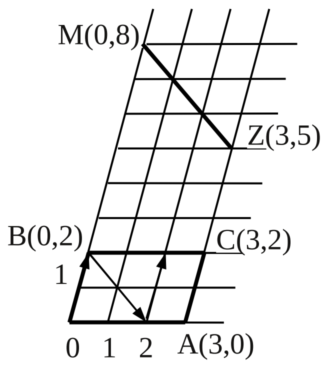

Fig. 5

| | | | | |
|---|---|---|---|---|
| 3 l. vessel | 0 | 0 | 2 | 2 |
| 2 l. vessel | 0 | 2 | 0 | 2 |
| 8 l. vessel | 8 | 6 | 6 | 4 |
| pour | start | 1st | 2nd | 3rd |

Table 10

**Answer.** 3 pours (the Case II solution)

**Example 3.** A vessel that is the biggest of the three available is liquid full, the other two are empty. The aim is to divide the liquid into two equal amounts, using only the three vessels with relevant volumes of 5, 8 and 12 liters. How would this happen? [2]

**Solution.** The geometrical way to solve the problem is presented by a drawing that shows the initial pouring ray move along either side OA or OB of OACDMB shape with its basal point at O. The construction sequence is similar to the previous problems. The drawing reading of the amount in the biggest vessel (the 12 liter one) is itself a triangle DBM, next to the basic figure OADCB as a supplement to form OACDM figure. The number of intersections along the basic figure's sides between pouring ray's reflection points and line MD of the supplement figure define the amount of liquid in the great vessel.

**Case I.** Ray starts along side OA.

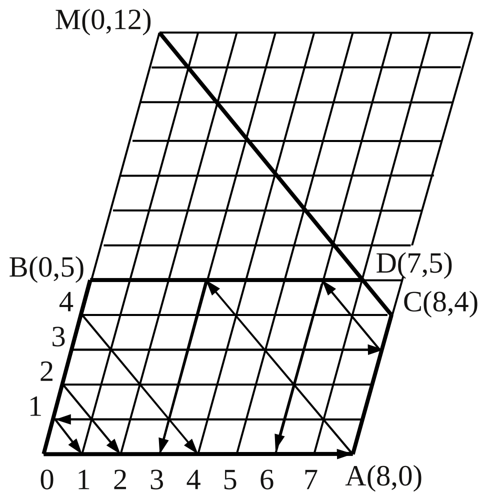


Fig. 6

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8-gal. vessel | 0 | 8 | 3 | 3 | 0 | 8 | 6 | 6 |
| 5-gal. vessel | 0 | 0 | 5 | 0 | 3 | 3 | 5 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 12-gal. vessel | 12 | 4 | 4 | 9 | 9 | 1 | 1 | 6 |
| pour | start | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th |

Table 11

**Case II.** Ray starts along side OB

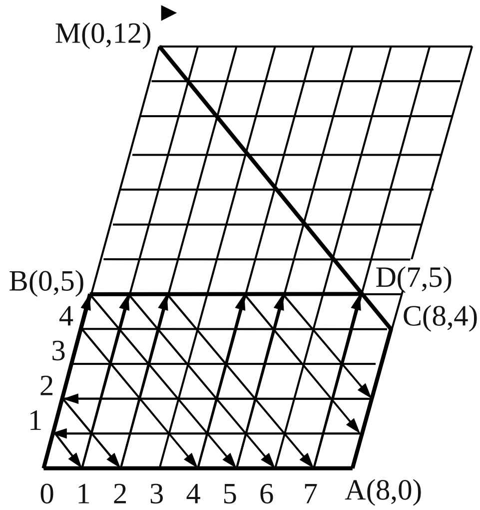


Fig. 7

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 l. | 0 | 0 | 5 | 5 | 8 | 0 | 2 | 2 | 7 | 7 | 8 | 0 | 4 | 5 | 8 | 0 | 1 | 1 | 6 |
| 5 l. | 0 | 5 | 0 | 5 | 2 | 2 | 0 | 5 | 0 | 5 | 4 | 4 | 0 | 4 | 1 | 1 | 0 | 5 | 0 |
| 12 l. | 12 | 7 | 7 | 2 | 2 | 10 | 10 | 5 | 5 | 0 | 0 | 8 | 8 | 3 | 3 | 11 | 11 | 6 | 6 |
| pour | start | | | | | | | | | | | | | | | | | | |

Table12

**Answer.** When the pouring ray begins along axis OA of the coordinate system, the pours are 7.

All the tables make clear that from the vessel with the most volume there are numerous ways of pouring an indefinite amount of liquid and different count of pours. However, is it always the case?

Let us consider the following problem: 4 liters of milk have to be poured out of a full 8 liter bucket. There are two other empty buckets available, a 6-liter and a 3-liter one.

If the pouring ray method is used, after the fourth reflection we discover that the ray comes back to point O, the basis of the coordinate system.

**Case 1.** Pouring ray starts along side OA.

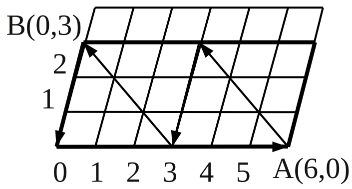


Fig. 8

| | | | | | |
|---|---|---|---|---|---|
| q6-l. | 0 | 6 | 3 | 3 | 0 |
| 3-l. | 0 | 0 | 3 | 0 | 3 |
| 8-l. | 8 | 2 | 2 | 5 | 5 |
| pour | start | 1st | 2nd | 3rd | 4th |

Table 13

**Case II.** Pouring ray starts along side OB.

| | | | | | |
|---|---|---|---|---|---|
| 6-l. | 0 | 0 | 3 | 3 | 6 |
| 3-l. | 0 | 3 | 0 | 3 | 0 |
| 8-l. | 8 | 5 | 5 | 2 | 2 |
| pour | start | 1st | 2nd | 3rd | 4th |

Table 14

Is this manner, if the pouring ray starts its motion along either OA or OB of the main figure, the reflection points are 4 in both cases. The tables and geometrical model show that there is no solution.

## CONCLUSION

Perelman’s geometrical method is an interesting and entertaining way of solving liquid pouring problems. Despite its application to one kind of problems only, when the biggest of three vessels $c$ is initially full and the other two $a$, $b$ are empty where $c \geq a + b$, this method presents an algorithm ensuring quick and accurate solution. Its essence does not require specific geometry knowledge and it is easy and accessible to learn and apply by students of different school grades.

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