SUPPORT STUDENTS
IN CREATING NEW MATHEMATICAL PROBLEMS

Emiliya Velikova
Department of Algebra and Geometry, Faculty of Natural Sciences and Education,
University of Ruse, 8 Studentska St., 7017 Ruse, Bulgaria, www.ru.acada.bg
emily@ami.ru.acad.bg

ABSTRACT

Promoting mathematical creativity for all students is an important part in finding new ways for the training of students, developing their creative potential not simply as consumers of knowledge but also for future successful scientific research or lifelong learning that will be of benefit to society.
In this respect the paper presents some mathematical methods and examples of CAS usage that support secondary school students in creating new mathematical problems.
Key words: *creativity, creating, mathematical problems, Computer Algebra Systems (CAS), Maple*

INTRODUCTION

From a historical point of view, almost all education systems oriented their efforts to students who could easily and quickly acquire educational material and apply it to education problems similar to those they study. In other words, school created consumers of ready knowledge. However, the speedy development of computer, information and communication technology at the end of the 20^{th} century brought us to the transition towards post-industrial information society, such as the societies of the USA and Japan in whose economies the production of commodities has been replaced by the production of services where information and knowledge stay as dominant production resources (www.iis.ru/library/riss/riss.ru.htm). The most valuable qualities for the labour market are the high level of education, professionalism, creativity, the opportunities for self-education and life-long learning. Hence, contemporary school was led to conclude that the skills for acquiring knowledge and their appropriate application are not enough for today's needs of society. A lot more treasured is the development of skills for independent acquisition of knowledge, as well as the ability to generate new knowledge. Some of the main goals of mathematics training into the twenty-first century are:

- ✓ Forming interest in Mathematics.
- ✓ Developing mathematical thinking.
- ✓ Developing students' mathematical creativity as:
  - Creative problem solving;
  - Creating new mathematical problems;
  - Generating multiple solutions;
  - Raising and proving the correctness of hypothesis, experimenting with facts, constructing a model on the basis of results, etc.
- ✓ Developing students' knowledge and skills for using computer, information and communication technologies.

CHARACTERISTICS OF THE MATHEMATICAL PROBLEMS USED AS A BASIS FOR STUDENTS' MATHEMATICAL CREATIVITY

This paper presents some methods for development and creating new problems on the basis of solved mathematical problems through:

- ✓ Transferring the problems from one mathematical area to another.
- ✓ Solving problems in a reversed order.
- ✓ Reformulating the mathematical problems (task).
- ✓ Changing or increasing the problems' (task's) conditions.
- ✓ Generalizing the mathematical problems, etc.

The basic mathematical problems are not always original but they ensure a wide range of opportunities for students (see the Romanian journal Gazeta Matematica: www.gazetamatematica.net, S.Grozdev, 2007) to:

- ✓ explore mathematical ideas (R.Millman, T.Jacobbe, 2008);
- ✓ formulate questions (L.Sheffield, 2003, p.147);
- ✓ construct many new examples (Watson & Mason, 2005);
- ✓ identify problem solving approaches that are useful for large classes of problems;
- ✓ look for "something more" (a generalization) in the mathematics they are working on;
- ✓ reflect on their results and find mistakes themselves;
- ✓ build new knowledge on the previous knowledge;
- ✓ discover previously unknown mathematical principles, concepts, and generalizations;
- ✓ think deeply (L.Sheffield, 2008);
- ✓ demonstrate abilities in a variety of ways, verbally, geometrically, graphically, algebraically, numerically;
- ✓ use technology such as Computer Algebra Systems as well as mathematical manipulatives and models, etc.

Other characteristics of the basic tasks might be (L.Sheffield, p.353-354):

- ✓ Existence of an entry point for all students which can be challenging even to the most advanced students.
- ✓ Connection to core standards and benchmarks in Mathematics.
- ✓ Interesting and should actively involve the students building a variety of thinking and learning styles.
- ✓ Open with more than one correct answer and/or more than one path to problem solution.

The teachers have to:

- ✓ ensure opportunities for students to explore, make mistakes, reflect, extend, and branch out into new related areas.
- ✓ give time for individual reflection and problems solving, as well as time for group exploration and discovery.

FIRST EXAMPLE

Problem 1. Solve the system

(1) $$\begin{cases} xz = y^2 \\ x^2 + y^2 = 25 \\ y^2 + z^2 = 600 \end{cases},$$

if x, y, z are real positive numbers.

Solution 1. From the first and the third and from the first and the second equations of (1) we obtain an equivalent system

$$\begin{cases} x(x+z) = 25 \\ z(z+x) = 600 \end{cases} \Rightarrow \frac{x}{z} = \frac{1}{24} \Rightarrow z = 24x.$$

It follows from (1) that $x = 1,\ y = 2\sqrt{6},\ z = 24$.

We lead students to a geometric solution.

Solution 2. Let ΔABC be a right triangle with $\measuredangle ACB = 90^0$, altitude $CD = y$ and $BC = a,\ AC = b,\ AD = x,\ DB = z$ (fig. 1). It follows from the Pythagorean theorem that $x^2 + y^2 = 25, y^2 + z^2 = 600$ for the triangles $ADC,\ BDC,$ respectively. And $xz = y^2$ because of the statement $AD.DB = DC^2$.

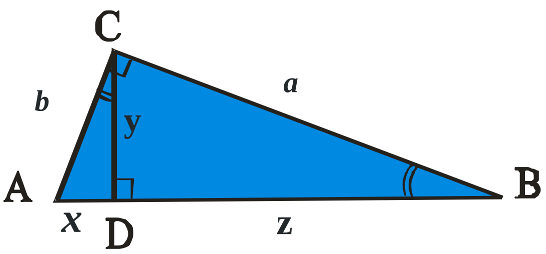

Figure 1

How we can support students in creating new mathematical problems on the basis of this given problem and its solution?

<u>Mathematical Method 1.</u> Creating a triangle ADC

<u>Discussion.</u> We discus with students that a system of x, y, z like (1) depends on creation of right triangles ABC, ADC, BDC. But the sides of the triangles BDC, ABC depend on the sides x, y of ΔADC. Because of the similarity $\Delta BDC, \Delta ADC$, i.e. $\frac{b}{x} = \frac{a}{y} \Rightarrow a = \frac{by}{x}$.

Let the sides of a right ΔABC are $x = 2\sqrt{2}, y = 2\sqrt{6}$. Then $b = \sqrt{x^2 + y^2} = 4\sqrt{2}$,

$z = \frac{y^2}{x} = 6\sqrt{2}$, $a = 4\sqrt{6} = \sqrt{y^2 + z^2}$. Then we formulate a new problem.

<u>Problem 2.</u> For any real positive numbers $x, y, z,$ solve the system

$$\begin{vmatrix} xz = y^2 \\ x^2 + y^2 = 32 \\ y^2 + z^2 = 96 \end{vmatrix} .$$

<u>Mathematical Method 2.</u> Reformulating the problem

It is obviously from the fig. 1 that $F_{\Delta ADC} + F_{\Delta BDC} = F_{\Delta ABC}$. Then we can calculate the volume of the expression $xy + yz = 2F_{\Delta ABC}$. We can reformulate the problem as a new one.

<u>Problem 3.</u> For any real positive numbers x, y, z, the following equations hold:

$$\begin{vmatrix} xz = y^2 \\ x^2 + y^2 = 32 \\ y^2 + z^2 = 96 \end{vmatrix} .$$

Determinate the value of $xy + yz$.

<u>Technological Method.</u> Using Computer Algebra System like Maple

The students can find many new problems by using Computer Algebra Systems, such as *Maple 11*. They have to choose aesthetic problems. Good examples are the following three problems:

```
> x:=2;y:=4*sqrt(2);b:=sqrt(x^2+y^2);z:=y^2/x;a:=b*y/x;
```

$$x := 2 \;\; y := 4\sqrt{2} \;\; b := 6 \;\; z := 16 \;\; a := 12\sqrt{2}$$

<u>Problem 4.</u> For any real positive numbers $x, y, z,$ solve the system

$$\begin{vmatrix} xz = y^2 \\ x^2 + y^2 = 36 \\ y^2 + z^2 = 188 \end{vmatrix} .$$

```
> x:=2;y:=2*sqrt(3);b:=sqrt(x^2+y^2);z:=y^2/x;a:=b*y/x;
```

$$x := 2 \;\; y := 2\sqrt{3} \;\; b := 4 \;\; z := 6 \;\; a := 4\sqrt{3}$$

Problem 5. For any real positive numbers x, y, z, the following equations hold:

$$\begin{vmatrix} xz = y^2 \\ x^2 + y^2 = 16 \\ y^2 + z^2 = 48 \end{vmatrix} .$$

Determinate the value of $xy + yz$.

```
> x:=12;y:=6;b:=sqrt(x^2+y^2);z:=y^2/x;a:=b*y/x;
```

$$x := 12 \;\; y := 6 \;\; b := 6\sqrt{5} \;\; z := 3 \;\; a := 3\sqrt{5}$$

Problem 6. For any real positive numbers x, y, z, the following equations hold:

$$\begin{vmatrix} xz = y^2 \\ x^2 + y^2 = 180 \\ y^2 + z^2 = 45 \end{vmatrix} .$$

Determinate the value of $xy + yz$.

A very bad example is the following problem

```
> x:=8;y:=15;b:=sqrt(x^2+y^2);z:=y^2/x;a:=b*y/x;
```

$$x := 8 \;\; y := 15 \;\; b := 17 \;\; z := \frac{225}{8} \;\; a := \frac{225}{8}$$

Problem 7. For any real positive numbers $x, y, z,$ solve the system

$$\begin{vmatrix} xz = y^2 \\ x^2 + y^2 = 289 \\ y^2 + z^2 = \dfrac{65025}{64} \end{vmatrix} .$$

SECOND EXAMPLE

Problem 8. For any right triangle ΔABC with sides a,b and c (a hypotenuse), prove the inequality

(2) $$a + b \le c\sqrt{2} .$$

Solution 1. It is given the right triangle ABC, $\measuredangle ACB = 90^0$ (fig. 2). We construct a segment $CD = b$ on the line BC. Then the triangle ABD, with side $BD = a + b$, is an acute triangle because $CA = CD = b$, i.e. $\measuredangle BDA = 45^0$. The distance between the point B and the line DA is BE and $BE \le AB$. It follows that $BE = BD.sin\,45^0 = (a+b).\dfrac{\sqrt{2}}{2} = \dfrac{a+b}{\sqrt{2}}$ because ΔBDE is a right triangle with altitude BE. Thus, $\frac{a+b}{\sqrt{2}} \leq c$. The equality occurs when $A \equiv E$. But $\measuredangle CAD = \measuredangle CED = 45^0$, i.e. $a = b$. Thus, the equality occurs if and only if $a = b$ (E.Velikova, 2008, p. 72).

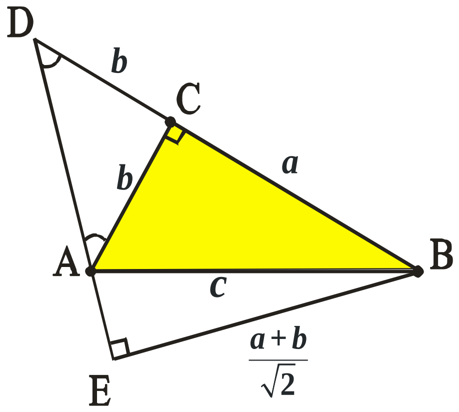

Figure 2

How we can support the mathematical creativity of the students on the basis of this problem?

<u>Discussion</u> on transfer the problem into geometric one.

If we construct a right triangle $ABC, \measuredangle ACB = 90^0$ with sides $a = BC = 1$, $b = AC = 2\sqrt{3}$ and a segment $CD = x$, such that $\measuredangle ACD = 30^0, \measuredangle BCD = 60^0$ (fig.3). Then $AD + DB = c$.

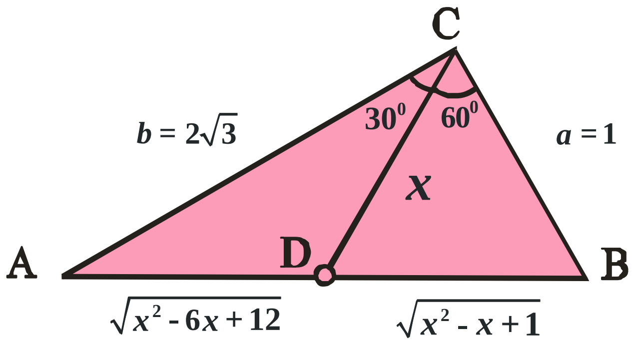

Figure 3

By the Cosine theorem for the sides AD, DB of $\Delta ACD, \Delta BCD$ we calculate

$$AD = \sqrt{x^2 - 2.x.2\sqrt{3}.cos\,30° + \left(2\sqrt{3}\right)^2} = \sqrt{x^2 - 6x + 12},$$

$$DB = \sqrt{x^2 - 2.x.1.cos\,60° + 1^2} = \sqrt{x^2 - x + 1}.$$

We use (2) $c > \dfrac{a+b}{\sqrt{2}}$ for $a \neq b$ and formulate the following problem.

Problem 9. For any real positive number x, the following inequality holds:

$$\sqrt{x^2-6x+12}+\sqrt{x^2-x+1} > \frac{\left(1+2\sqrt{3}\right)\sqrt{2}}{2}.$$

The students usually develop themselves many different methods for creating new problems. The new results depend on students' knowledge, abilities, skills, and interest.

Mathematical Method 1. ***Changing the length of the sides a,b***

Good Example is the following one.

Problem 10. For any real positive number x, the following inequality holds:

$$\sqrt{x^2-3x+3}+\sqrt{x^2-2x+4} > \frac{\left(2+\sqrt{3}\right)\sqrt{2}}{2}.$$

Remark. The geometric solution is based on the inequality (2), where $a=2, b=\sqrt{3}$.

Mathematical Method 2. ***Changing the angles $\measuredangle ACD, \measuredangle BCD$***

Problem 11. For any real positive number x, the following inequality holds:

$$\sqrt{x^2-2\left(\sqrt{3}+1\right)x+8}+\sqrt{x^2-\left(\sqrt{3}-1\right)x+2} > 3.$$

Remark. The geometric solution is based on the inequality (2), where $a=\sqrt{2}, b=2\sqrt{2}$ and $\measuredangle ACD=15^0, \measuredangle BCD=75^0$.

Mathematical Method 3. ***Changing the number of the angles***

Problem 12. For any real positive numbers x,y, the following inequality holds:

$$\sqrt{x^2-\left(\sqrt{3}+1\right)x+2}+\sqrt{x^2-\sqrt{3}xy+y^2}+\sqrt{y^2-4y+8} > 3.$$

Remark. The geometric solution is based on the inequality (2), where $a=2\sqrt{2}, b=\sqrt{2}$ and $\measuredangle ACD=15^0, \measuredangle DCE=30^0, \measuredangle ECB=45^0$ (fig. 4).

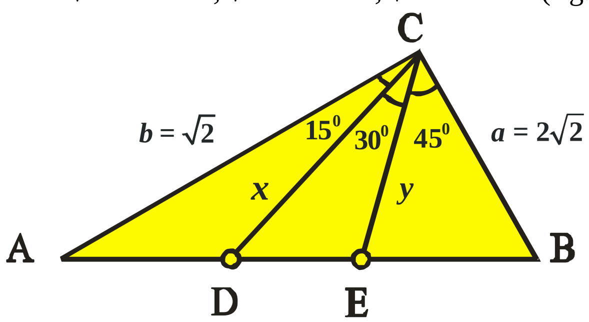

Figure 4

Mathematical Method 4. ***Generalization of the problem***

Problem 13. For any real positive numbers $x, y \neq z$, the following inequality holds:

$$\sqrt{x^2 - xy + y^2} + \sqrt{x^2 - \sqrt{3}xz + z^2} > \frac{y+z}{\sqrt{2}}.$$

Remark. The geometric solution is based on the inequality (2) where $a = z, b = y$ and $\measuredangle ACD = 60^0, \measuredangle BCD = 30^0$.

Technological Method. ***Using Computer Algebra System***

With the students we prepare a program for calculating inequalities which supports students' ideas in creating new problems and to chose the best ones:

```
> a:=5;b:=3;CD:=x;alpha:=Pi/6;beta:=Pi/3;
evalm(sqrt(CD^2-2*CD*b*cos(alpha)+b^2)+
sqrt(CD^2-2*CD*a*cos(beta)+a^2)>(a+b)/sqrt(2));
```

$$a := 5 \quad b := 3 \quad CD := x \quad \alpha := \frac{1}{6}\pi \quad \beta := \frac{1}{3}\pi \quad 4\sqrt{2} < \sqrt{x^2 - 3x\sqrt{3} + 9} + \sqrt{x^2 - 5x + 25}$$

The students formulate a new problem.

Problem 14. For any real positive number x, the following inequality holds:

$$\sqrt{x^2 - 3x\sqrt{3} + 9} + \sqrt{x^2 - 5x + 25} > 4\sqrt{2}.$$

```
> a:=m;b:=n;CD:=x;alpha:=Pi/6;beta:=Pi/3;
evalm(sqrt(CD^2-2*CD*b*cos(alpha)+b^2)+
sqrt(CD^2-2*CD*a*cos(beta)+a^2)>(a+b)/sqrt(2));
```

Problem 15. For any real positive numbers x, m, n, the following inequality holds:

$$\sqrt{x^2 - nx\sqrt{3} + n^2} + \sqrt{x^2 - mx + m^2} > \frac{1}{2}\sqrt{2}(n+m)$$

THIRD EXAMPLE

For the sides of a triangle ABC, the **"Basic triangle inequality"** holds:

(3) $AB + BC \geq AC$

It is very useful as a basis for students to create new problems (fig. 5).

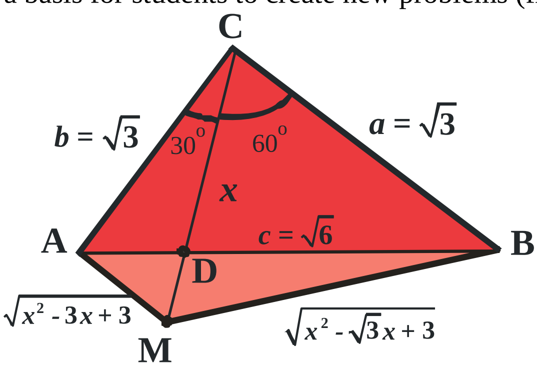

Figure 5

Problem 14. Prove that for any real number x, the following inequality holds:

$$\sqrt{x^2-3x+3}+\sqrt{x^2-\sqrt{3}x+3}\geq\sqrt{6}\,.$$

The students use methods which are presented above like: reformulating, changing values of sides, angles, number of sides, etc.

Problem 15. Find the best low bound of the function:

$$f(x)=\sqrt{x^2-3x+3}+\sqrt{x^2-\sqrt{3}x+3}\,.$$

Problem 16. Prove that for any real number x, the following inequality holds:

$$\sqrt{4x^2-2x+1}+\sqrt{x^2-x+1}+x\sqrt{3}\geq 2 \text{ (fig. 6).}$$

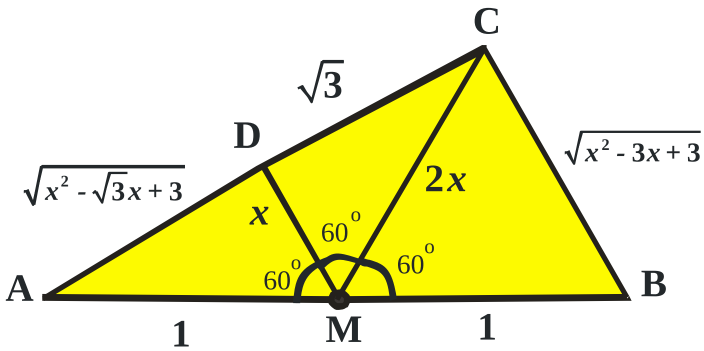

Figure 6

Problem 17. Prove that for any real number x, the following inequality holds:

$$\sqrt{x^2-\sqrt{3}x+1}+\sqrt{x^2-xy+y^2}+\sqrt{y^2-\sqrt{3}y+1}\geq\sqrt{3} \text{ (fig. 7).}$$

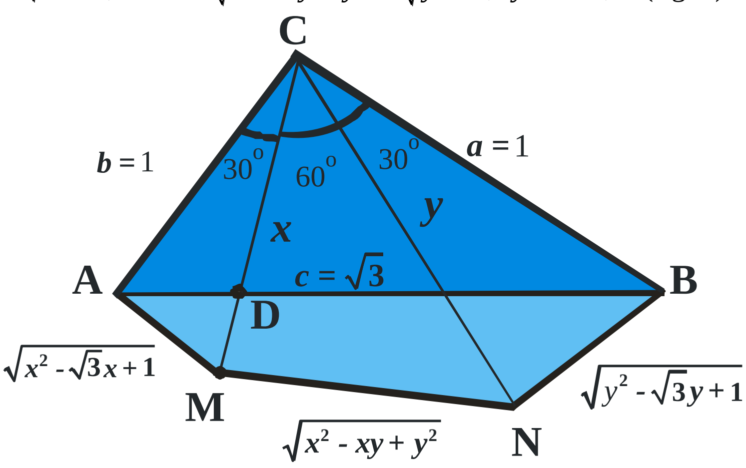

Figure 7

CONCLUSION

The support of all students for creating new mathematical knowledge through different representations, connections between different concepts and transfer of problems from one mathematical area to another and on the basis of CAS is very important training process in Mathematics which develops the students as creators of knowledge and as persons of our future.

REFERENCES

Grozdev, S. For High Achievements in Mathematics. The Bulgarian Experience (Theory and Practice). Assoc. for the Development of Education, Sofia, 2007. 295 p.

Millman, R. & T. Jacobbe (2008) Fostering Creativity in Preserves Teachers Through Mathematical Habits of the Mind, In Promoting Creativity for All Students in Mathematics Education, E.Velikova, A.Andzans (Eds.), Proceedings of the DG 9: Promoting Creativity for All Students in Mathematics Education, ICME 11, Monterrey, Mexico, July 6-13, 2008, University of Ruse, University of Latvia, p.146-151.

Sheffield, L. (2008) Promoting Creativity for All Students in Mathematics Education: An Overview, In Promoting Creativity for All Students in Mathematics Education, E.Velikova, A.Andzans (Eds.), Proceedings of the DG 9: Promoting Creativity for All Students in Mathematics Education, ICME 11, Monterrey, Mexico, July 6-13, 2008, University of Ruse, University of Latvia, p.346-358.

Sheffield, L. J. (2003). *Extending the challenge in Mathematics: Developing mathematical promise in K - 8 pupils*. Thousand Oaks, CA: Corwin Press.

Velikova, E. (2008) The Beauty of the Geometric Inequalities, Multiple Solution Connecting Tasks, Proceedings of the International Research Workshop of the Israel Science Foundation, 20-22 Feb., 2008, Haifa, Israel, p.61-74.

Watson, A. & J.Mason, (2005). *Mathematics as a Constructive Activity: Learners Generating Examples*, Lawrence Erlbaum Associates Publishers.