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ON AN APPROACH FOR CONSTRUCTION OF I-V CURVES IN LJJ

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Abstract. New approach for construction of current-voltage (I-V) curves which are significant in Physics of long Josephson junctions (LJJ) is presented in this paper. The bifurcational static solution and eigenfunctions of the corresponding Sturm-Liouville problem is used. After discretization an overdetermined nonlinear algebraic system is obtained which is solved by least square method. The properties of eigenfunctions lead to the possibility to solve the problem recursively using the continuous analog of Newton's method. An algorithm for successive finding of I-V curve points is described.

Key words: long Josephson junction, Sturm-Liouville, continuous analog of Newton's method, bifurcation, sine-Gordon, I-V curve

Mathematics Subject Classification 2000: 33F05, 34B15, 34B24, 34K28, 34L16, 74K30

1. Introduction

Here we solve the dynamical problem concerning the magnetic flux in an one-layer homogeneous LJJ which is simulated by the equation [1]

$$\phi_{xx} - \phi_{tt} - \alpha\phi_t = \sin\phi - \gamma\,, \quad t > 0\,, \quad x \in (-l, l) \tag{1}$$

with boundary conditions

$$\phi_x(\pm l, t) = h_e\,, \tag{2}$$

where h_e is an external Josephson junction magnetic field. Let us suppose that at $t = 0$ the dynamical solution coincides with the corresponding static solution which is denoted by $\varphi(x)$. This is a solution of the following static problem:

$$-\varphi_{xx} + \sin\varphi - \gamma = 0 \tag{3}$$

with boundary conditions

$$\varphi_x(\pm l) = h_e\,.$$

Of course, this is the same problem as the dynamical one but without the derivatives with respect to time.

The following Sturm-Liouville problem (StLP) corresponds to this problem

$$-\psi_{xx} + q(x)\psi = \lambda\psi\,, \tag{4}$$

$$\psi_x(\pm l) = 0\,, \tag{5}$$

under norming condition

$$\int_{-l}^{l} \psi^2\,dx - 1 = 0\,, \tag{6}$$

where the weight $q(x) = \cos\varphi(x)$.

In order to find the static solution and to use it as an initial approximation for the dynamic one first of all the bifurcation problem must be solved, i.e. it is assumed that $\lambda = 0$ or λ is sufficiently small, for example $\lambda = 10^{-5}$. This means that the static problem and StLP are solved simultaneously at $\lambda = 10^{-5}$. So we have the model

$$-\varphi_{b,xx} + \sin\varphi_b - \gamma_{cr} = 0\,, \tag{7a}$$

$$\varphi_{b,x}(\pm l) - h_e = 0\,, \tag{7b}$$

$$-\psi_{xx} + [q_b(x) - \lambda]\psi = 0\,, \tag{7c}$$

$$\psi_x(\pm l) = 0\,, \tag{7d}$$

$$\int_{-l}^{l} \psi^2\,dx - 1 = 0\,. \tag{7e}$$

Further the received solutions $\varphi_b(x)$ and $\psi_n(x)$, $n = 1, \ldots, N$ are used for the construction of the dynamical distribution as follows

$$\tilde{\phi}(x,t) = \varphi_b(x) + e^{-\alpha t/2}\sum_{n=1}^{N}\xi_n(t)\psi_n(x)\,. \tag{8}$$

For the sake of simplicity the counting of n is begun with 1. Of course this counting can not coincide with the numbers of eigenvalues $\psi_n(x)$.

In order to find the required dynamical distribution the functions $\xi_n(t)$, $n = 1, \dots, N$ must be defined.

Using (8) in such form is proper because

(1) for the sake of choice $\varphi_b(x)$ and $\psi_n(x)$, $n = 1, \dots, N$ the boundary conditions (2) are satisfied;

(2) functions $\psi_n(x)$, $1 = 0, \dots, N$ are orthonormalized;

(3) dynamical distribution can be received by the "perturbation" of the bifurcation solution;

(4) the problem decomposes because only dynamical functions $\xi_n(t)$, $n = 1, \dots, N$ are sought but space functions $\psi_n(x)$, $n = 1, \dots, N$ remain the same as found in advance;

(5) due to the exponent $e^{-\alpha t/2}$ the coefficient in front of the first derivative with respect to time in (1) is zero (for details see §2).

2. Insertion of an exponent into the decomposition for $\phi(x,t)$

The solution of the problem (1) is searched in the decomposed form

$$\phi(x,t) = \varphi_b(x) + f(t) \sum_{n=1}^{N} \xi_n(t)\psi_n(x) \,. \tag{9}$$

where the function $f(t)$ is found (9) thus, the derivatives $\xi_{n,t}(t)$, $n = 1, \dots, N$ do not take part in the final form of the equation after the insertion of (9) into (1).

Initially, we find the first and second derivatives of the function $\phi(x,t)$ with respect to time t and to space coordinate x $\phi_t(x,t)$, $\phi_{tt}(x,t)$, $\phi_x(x,t)$ and

$\phi_{xx}(x,t)$. After that we put their expressions into the equation (1) and receive

$$\varphi_{b,xx}(x)+f(t)\sum_{n=1}^{N}\xi_n(t)\psi_{n,xx}(x)-f_{tt}(t)\sum_{n=1}^{N}\xi_n(t)\psi_n(x)-2f_t(t)\sum_{n=1}^{N}\xi_{n,t}(t)\psi_n(x)-$$
$$-f(t)\sum_{n=1}^{N}\xi_{n,tt}(t)\psi_n(x)-\alpha f_t(t)\sum_{n=1}^{N}\xi_n(t)\psi_n(x)-\alpha f(t)\sum_{n=1}^{N}\xi_{n,t}(t)\psi_n(x)=$$
$$=\sin\phi(x,t)-\gamma$$

Equating the coefficient in front of the derivative $\xi_{n,t}(t)$ to zero i.e.

$$-2f_t(t)-\alpha f(t)=0\,.$$

and solving this differential equation we obtain

$$f(t)=e^{-\alpha t/2}\,.$$

Thus, it is shown that the choice of function $f(t)=e^{-\alpha t/2}$ in the expression (9) helps us work further without first derivative $\xi_n(t)$ with respect to time.

3. Discretization

First of all our aim is to find values of the required function in points of some grid of interval $[0,T]$, where T is the upper bound of the considered time interval, which is defined during the work i.e. we need to fill a table of values $\xi_n(t_j)$, $n=1,\dots,N$, $\tau=T/M$, $t_j=j\tau$, $j=\overline{0,M}$. In order to form two-dimensional grid (x_i,t_j) we use the values of x_i from the grid of the mixed static problem (7). We calculate the derivatives $\phi_t(x,t)$, $\phi_{tt}(x,t)$, $\phi_x(x,t)$, $\phi_{xx}(x,t)$, put them into the expression (1) and obtain that the equation

$$\varphi_{b,xx}(x_i)+E_j\sum_{n=1}^{N}\left[\xi_n(t_j)\psi_{n,xx}(x_i)+\left(\frac{\alpha^2}{4}\xi_n(t_j)-\xi_{n,tt}(t_j)\right)\psi_n(x_i)\right]-$$
$$-\sin\phi(x_i,t_j)+\gamma=0\,, \tag{10}$$

where

$$E_j=e^{-\alpha t_j/2}$$

and

$$\phi(x_i,t_j)=\varphi_b(x_i)+E_j\sum_{n=0}^{N}\xi_n(t_j)\psi_n(x_i)$$

must be fulfilled in every point (x_i, t_j).

Since the functions $\varphi_b(x)$ and $\psi_n(x)$, $n = 1, \ldots, N$ in grid knots are already found we use their values in points x_i and for the second derivatives in (10) we can make use of the following formulae which are received from the equations (3) and (4):

$$\varphi_{b,xx}(x_i) = \sin\varphi_b(x_i) - \gamma_{cr}\,, \quad i = 0, \ldots, K \tag{11}$$

$$\psi_{n,xx} = [q(x_i) - \lambda_n]\psi_n(x_i)\,, \quad i = 0, \ldots, K \tag{12}$$

Further, we replace the values $\xi_{n,tt}$, $n = 0, \ldots, N$ by the formulae [2]

$$\xi_{n,tt}(t_j) = \frac{\xi_n(t_{j+1}) - 2\xi_n(t_j) + \xi_n(t_{j-1})}{\tau^2}\,, \quad j = 1, \ldots, M\,. \tag{13}$$

Since at $t = t_0 = 0$ the condition

$$\phi(x, 0) = \varphi_b(x) \tag{14}$$

must be fulfilled, it follows that for the required functions $\xi_n(t)$, $n = 1, \ldots, N$ the following condition

$$\sum_{n=1}^{N} \xi_n(0)\psi_n(x) = 0 \tag{15}$$

must be fulfilled.

Since the functions $\psi_n(x)$ are linearly independent, then the condition (15) is fulfilled if and only if zero initial conditions for required functions

$$\xi_n(0) = 0\,, \quad n = 1, \ldots, N \tag{16}$$

are fulfilled.

If we replace in (10) all the derivatives by the formulae (11), (12) and (13) for every $j = 1, \ldots, M$ we receive $K+1$ equations for every $i = 0, \ldots, K$. The equality (10) can be written in the following way

$$f_{i,j}[\xi_0(t_{j-1}), \xi_0(t_j), \xi_0(t_{j+1}), \ldots, \xi_N(t_{j-1}), \xi_N(t_j), \xi_N(t_{j+1})] = 0 \tag{17}$$

for every $j = 1, \ldots, M \;\; i = 0, \ldots, K$, where

$$f_{i,j} = \varphi_{b,xx}(x_i) +$$

$$+E_j \sum_{n=1}^{N} \left[\xi_n(t_j)\psi_{n,xx}(x_i) + \left(\frac{\alpha^2}{4}\xi_n(t_j) - \frac{\xi_n(t_{j+1}) - 2\xi_n(t_j) + \xi_n(t_{j-1})}{\tau^2}\right)\psi_n(x_i)\right] -$$

$$-\sin\phi(x_i, t_j) + \gamma\,, \quad i = 0, \ldots, K\,, \quad j = 1, \ldots, M \tag{18}$$

and the second derivatives $\varphi_{b,xx}(x_i)$ and $\psi_{n,xx}(x_i)$ are defined by (11) and (12).

At $j = 0$ we have

$$\xi_n(t_0) = 0\,, \quad n = 0, \ldots, N\,. \tag{19}$$

Thus at every step $j = 1, \ldots, M$ we receive an overdetermined system for unknown values $\xi_n(t_{j-1})$, $\xi_n(t_j)$, $\xi_n(t_{j+1})$, $n = 0, \ldots, N$. Such a system can be solved by least square method [2] i.e. the problem considered reduces itself to minimum finding of the function

$$\mathcal{F}_j = \sum_{i=0}^{K} f_{i,j}^2\,, \quad j = 0, \ldots, M\,. \tag{20}$$

Further in order to get some important simplification, the sum (20) will begin from $j = 1$. Since the function $\mathcal{F}_j$ (20) is positively definite, the necessary and sufficient conditions for minimum of $\mathcal{F}_j$ are

$$\frac{\partial \mathcal{F}_j}{\partial \xi_k(t_{j-1})} = 0\,, \quad k = 1, \ldots, N\,, \tag{21a}$$

$$\frac{\partial \mathcal{F}_j}{\partial \xi_k(t_j)} = 0\,, \quad k = 1, \ldots, N\,, \tag{21b}$$

$$\frac{\partial \mathcal{F}_j}{\partial \xi_k(t_{j+1})} = 0\,, \quad k = 1, \ldots, N \tag{21c}$$

for every $j \geq 1$.

These equations can be written in the following way. The first group of equations (21a) and the last one (21c) coincide:

$$\sum_{i=1}^{K} f_{i,j}\, \psi_k(x_i) = 0\,, \quad k = 1, \ldots, N\,, \quad \forall j \geq 1 \tag{22}$$

and the second group (21b) looks as follows:

$$\sum_{i=1}^{K} f_{i,j}\, \left[\psi_{k,xx}(x_i) - \cos\phi(x_i, t_j)\, \psi_k(x_i)\right] = 0\,, \quad k = 1, \ldots, N\,, \quad \forall j \geq 1\,, \tag{23}$$

where $f_{i,j}$ is given by the expression (18). Firstly, let us pay attention to the first group equations. They look in details as

$$\sum_{i=1}^{K}\left\{\varphi_{b,xx}(x_i)+E_j\sum_{n=1}^{N}\left[\xi_n(t_j)\psi_{n,xx}(x_i)+\right.\right. \tag{24}$$

$$\left.\left.\left(\frac{\alpha^2}{4}\xi_n(t_j)-\frac{\xi_n(t_{j+1})-2\xi_n(t_j)+\xi_n(t_{j-1})}{\tau^2}\right)\psi_n(x_i)\right]-\sin\phi(x_i,t_j)+\gamma\right\}\psi_k(x_i)=0\,,$$

$$k=1,\ldots,N\,,\quad \forall j\geq 1\,.$$

Let us note that we can use the orthonormality condition for functions $\psi_n(x)$, $n=1,\ldots,N$ i.e.

$$\int_{-l}^{l}\psi_n(x)\psi_k(x)\,dx=\delta_{nk}\,,\quad n=1,\ldots,N\,,\quad k=1,\ldots,N$$

or approximately

$$\sum_{i=1}^{K}\psi_n(x)\psi_k(x)=\frac{1}{h}\,\delta_{kn}\,,\quad n=1,\ldots,N\,,\quad k=1,\ldots,N\,, \tag{25}$$

where δ_{nk} is a Kronecker delta.

For the sake of convenience we denote the coefficient in front of $\psi_n(x_i)$ in (24) by S_{nj} i.e.

$$S_{nj}=\frac{\alpha^2}{4}\,\xi_n(t_j)-\xi_{n,tt}(t_j)\,.$$

Then the system (24) can be written in the following way

$$\sum_{i=1}^{K}\left[\varphi_{b,xx}(x_i)+E_j\sum_{n=1}^{N}\xi_n(t_j)\psi_{n,xx}(x_i)\;-\sin\phi(x_i,t_j)+\gamma\right]\psi_k(x_i)+ \tag{26}$$

$$+E_j\sum_{n=1}^{N}S_{nj}\sum_{i=1}^{K}\psi_n(x)\psi_k(x)=0\,,\quad k=1,\ldots,N\,,\quad \forall j\geq 1\,.$$

Using (25) we can express S_{kj} as follows

$$S_{kj}=-\frac{h}{E_j}\sum_{i=1}^{K}\left[\varphi_{b,xx}(x_i)+E_j\sum_{n=1}^{N}\xi_n(t_j)\psi_{n,xx}(x_i)-\sin\phi(x_i,t_j)+\gamma\right]\psi_k(x_i)\,, \tag{27}$$

$$k=1,\ldots,N\,,\quad \forall j\geq 1\,.$$

Let us note that the expressions S_{kj}, $k = 1, \ldots, N$ take parts in f_{ij} and if in (23) we replace them by () then the unknowns $\xi_k(t_{j-1}), \xi_k(t_{j+1})$, $k = 1, \ldots, N$ drop out. Thus, only a system with respect to unknowns $\xi_k(t_j)$, $k = 1, \ldots, N$ which looks as follows

$$\sum_{i=1}^{K} \left\{ \varphi_{b,xx}(x_i) + E_j \sum_{n=1}^{N} [\xi_n(t_j)\psi_{n,xx}(x_i) \; + S_{nj}\, \psi_n(x_i)] - \sin\phi(x_i, t_j) + \gamma \right\} \times$$
$$\times \, [\psi_{k,xx}(x_i) - \cos\phi(x_i, t_j)\, \psi_k(x_i)] = 0\,, \quad k = 1, \ldots, N\,, \quad \forall j \geq 1 \tag{28}$$

remains.

We solve the system (28) by continuous analog of Newton's method [3]. Since the initial approximations are $\xi_n(t_0) = 0$ then due to the zero initial conditions for the first derivative, it follows that $\xi_n(t_1) = 0\,, n = 1, \ldots, N$ with an accuracy $O(\tau)$. Therefore we begin with $\xi_n(t_1) = 0\,, n = 1, \ldots, N$ as initial approximations for the system (28). We use the received solutions as initial approximations for the system at the next time layer and the voltage $V(t_j)$ is calculated. The calculation of averaged voltage can be performed by the formula

$$V = \frac{1}{t_j}\frac{1}{2l} \int_0^{t_j} \int_{-l}^{l} \phi_t(x, t)\, dxdt\,. \tag{29}$$

After substitution (8) into (29) we receive

$$V = \frac{1}{2t_j l} \sum_{n=0}^{N} A_n e^{-\alpha t_j/2} \xi_n(t_j)\,, \tag{30}$$

where

$$A_n = \int_{-l}^{l} \psi_n\, dx\,.$$

The passage to the next layer stops if

$$|V(t_{j+1}) - V(t_j)| < \delta V\,,$$

where δV is a beforehand given small number. Thus we reach some stabilization of the described process.

4. Algorithm

Finally, the following scheme for construction of current-voltage characteristic is proposed here:

(1) set physical parameters (l, h_e α);

(2) solve a static problem at $\lambda = 10^{-5}$ (for example) with a spectral parameter γ and receive φ_b, ψ_n, $n = 0, \ldots, N$ and $\gamma = \gamma_{cr}$;

(3) begin with γ_{cr} and decrease it by some step $\delta\gamma$ until reaching the condition $V = 0$ and, besides, increasing it to some value (for example $\gamma = 1$);

(4) find $\xi_n(t_j)$, $n = 0, \ldots, N$ at the layer j by the described in item (3) way at the corresponding value of external current γ;

- Note 1. At $\gamma = \gamma_{cr}$ we receive zero solutions i.e. all the required functions are identically equal to zero $\xi_n(t_j) = 0\,, n = 1, \ldots, N\,, j = 0, \ldots, M$. This must be in view of the fact that at this point we have a bifurcational solution which is already obtained by solving the static problem.
- Note 2. At an external current value less than the critical one ($\delta\gamma < 0$) zero voltages ($V(t_M) = 0$) must be received.

(5) calculate V by the formula (30); $j = j+1$ (the passage to the next layer t_j) and make a passage to item (4) until the condition $|V(t_{j+1}) - V(t_j)| < \delta V$ is fulfilled. The last voltage obtained V and the considered current γ define a point (γ, V) from the current-voltage (I-V) curve;

(6) go to item (3) until the exhausting of the γ values;

We plan to announce and discuss numerical results of the described algorithm in the next communication.

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ВЪРХУ ЕДИН ПОДХОД ЗА КОНСТРУИРАНЕ НА I-V КРИВИ НА ДДК

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Резюме. В тази работа е предложен нов подход за построяване на волт-амперни (I-V) криви, които са особено важни във физиката на дългите джозефсонови контакти (ДДК). Използва се бифуркационното статично решение и собствените функции на съответната задача на Щурм-Лиувил. След дискретизацията се получава преопределена нелинейна алгебрична система, за която се предлага да се решава по метода на най-малките квадрати. Свойствата на собствените функции довежда до възможността задачата да се решава рекурсивно с използването на непрекъснатия аналог на метода на Нютон. Описан е алгоритъмът за последователно намиране на точките от I-V кривата.