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FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH COMMUTING HIGHER ORDER JACOBI OPERATORS

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Abstract. We consider four-dimensional Riemannian manifolds with commuting higher order Jacobi operators defined on two-dimensional orthogonal subspaces (polygons) and on their orthogonal subspaces.

More precisely, we discuss higher order Jacobi operator $\mathcal{J}(X)$ and its commuting associated operator $\mathcal{J}(X^{\perp})$ induced by the orthogonal complement $X^{\perp}$ of the vector X, i. e. $\mathcal{J}(X) \circ \mathcal{J}(X^{\perp}) = \mathcal{J}(X^{\perp}) \circ \mathcal{J}(X)$.

At the end some new central theorems have been cited. The latter are due to P. Gilkey, E. Puffini and V. Videv, and have been recently obtained.

Key words: Einstein manifold, Higher order Jacobi operator, Jacobi operator, Ricci operator

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1. Preliminaries

Let (M, g) be a n-dimensional Riemannian manifold with a metric tensor g. Tangent space at a point $p \in M$ we denote by M_p, and let S_pM be the set of unit vectors in M_p, i. e. $S_p(M) := \{z \in M_p \mid \; \|g(z,z)\| = 1\}$. Let $\mathcal{F}(M)$ be the algebra of all smooth functions on M and $\mathcal{X}(M)$ be the $\mathcal{F}(M)$-module of all smooth vector fields over M. Let also

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad X, Y, Z \in \mathcal{X}(M)$$

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be the $(1,3)$ curvature tensor of the Levi-Civita connection ∇. We define

$$R(X,Y,Z,U) := g(R(X,Y,Z),U)$$

to be the associated $(0,4)$-curvature tensor which satisfied the following algebraic properties:

i) $R(X,Y,Z,U) = -R(Y,X,Z,U)$,
ii) $R(X,Y,Z,U) = -R(X,Y,U,Z)$,
iii) $R(X,Y,Z,U) + R(Y,Z,X,U) + R(Z,X,Y,U) = 0$ (first Bianchi identity),
iv) $R(X,Y,Z,U) = R(Z,U,X,Y)$.

In the Riemannian geometry the following differential equality is also true:

v) $(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0$

(second Bianchi identity), where

$$(\nabla_X R)(Y,Z,W) := \nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W,$$

and $\nabla_X R$ is the covariant derivative of the $(1,3)$-curvature tensor R with respect to X, $X,Y,Z \in \mathcal{X}(M)$.

Let $\mathcal{J}\colon M_p \longrightarrow M_p$ be the Jacobi operator defined by:

$$\mathcal{J}(X)U = R(U,X,X). \tag{1.1}$$

One can easily see that $\mathcal{J}(X)X = 0$ and $g(\mathcal{J}(X)Y,Z) = g(Y,\mathcal{J}(X)Z)$ which means that Jacobi operator is a symmetric linear operator.

Jacobi operator can be diagonalized in the Riemannian geometry. In this case we say that g is *Osserman metric* if the eigenvalues of the Jacobi operator are constant over the tangent bundle $S(M) := \bigcup_{p\in M} S_pM$. If (M,g) is a rank one locally symmetric space, i. e. $\nabla R = 0$, where ∇ is the connection with all positive or all negative sectional curvatures [14] or (M,g) is flat, i. e. $R = 0$, the group of local isometries acts transitively on $S(M)$ and each Jacobi operator has constant eigenvalues. Osserman [8] conjectured that the opposite is also true and this was confirmed by Chi when $n = 4$ and $n \equiv (2 \mod 4)$ [2] and by Nikolaevsky [7] when $n \neq 16$.

Gilkey, Stanilov and Videv [5] introduced a new operator which they called *general Jacobi operator of order k or k-order Jacobi operator*. More precisely, if $\{Y_i\}_{i=1}^k$ is any orthonormal basis for an arbitrary k-plane $\pi \in M_p$, the higher k-order Jacobi operator is defined by:

$$\mathcal{J}(\pi)\colon Y \longrightarrow \sum_{1\le i\le k} R(Y,Y_i)Y_i = \sum_{1\le i\le k} \mathcal{J}(Y_i)Y. \tag{1.2}$$

It can be easily verified that this operator does not depend on the basis of π.

2. Some commutativity conditions

Another variety of problems, connected to the higher order Jacobi operator, emerged thanks to Stanilov and Videv [11]. They are connected with some commutativity conditions forced on (1.2). Recently Brozos-Vázquez and Gilkey [1] were able to prove the following

Theorem 2.1. *Let (M,g) be a Riemannian manifold,* $\dim M \ge 3$. *Then*

- *(A) (M,g) is flat iff $\mathcal{J}(X)\mathcal{J}(Y) = \mathcal{J}(Y)\mathcal{J}(X)$ for arbitrary vectors $X,Y\in M_p$;*
- *(B) (M,g) is a manifold with a constant sectional curvature iff $\mathcal{J}(X)\mathcal{J}(Y) = \mathcal{J}(Y)\mathcal{J}(X)$ for arbitrary vectors $X,Y \in M_p$ such that $X \perp Y$.*

In this paper authors will characterize four-dimensional Riemannian manifolds that satisfy the following two conditions:

For arbitrary unit vector $X \in M_p$, $p \in M$, we have:

(C1) $\mathcal{J}(X)\circ\mathcal{J}(X^\perp) = \mathcal{J}(X^\perp)\circ\mathcal{J}(X)$, *where $X^\perp$ is the orthogonal complement of X in M_p.*

For arbitrary 2-plane $\alpha \subset M_p$, $p \in M$, we have

(C2) $\mathcal{J}(\alpha)\circ\mathcal{J}(\alpha^\perp) = \mathcal{J}(\alpha^\perp)\circ\mathcal{J}(\alpha)$, *where $\alpha^\perp$ is the orthogonal complement of α in M_p.*

Our main goal is to prove the following

Theorem 2.2. *Let (M,g) be a four-dimensional Riemannian manifold. Then the following are equivalent:*

- *(a) Equality* **(C1)** *holds for arbitrary unit vector $X \in M_p$, $p \in M$;*
- *(b) Equality* **(C2)** *holds for arbitrary 2-plane $\alpha \subset M_p$, $p \in M_p$;*
- *(c) (M,g) is Einstein.*

Proof. (**a**) $\Longrightarrow$ (**c**) Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for M_p, $p \in M$. Curvature operator matrices $\mathcal{J}_{\{e_1,e_2,e_3\}}$ and $\mathcal{J}_{\{e_4\}}$ then have the form:

$$\begin{pmatrix} K_{12}+K_{13} & R_{1332} & R_{1223} & \rho_{14} \\ R_{1332} & K_{12}+K_{13} & R_{2113} & \rho_{24} \\ R_{1223} & R_{2113} & K_{13}+K_{23} & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & K_{14}+K_{24}+K_{34} \end{pmatrix}, \tag{2.3}$$

and

$$\begin{pmatrix} K_{14} & R_{1442} & R_{1443} & 0 \\ R_{1442} & K_{24} & R_{2443} & 0 \\ R_{1443} & R_{2443} & K_{34} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

where $\rho_{ij} = \rho(e_i, e_j) := \sum_{k=1}^{4} g(R(e_k, e_i)e_j, e_k)$ are the components of the Ricci $(0,2)$-tensor ρ and $K_{ij} := g(R(e_i, e_j)e_j, e_i)$, $i, j = 1, \ldots, 4$.

We have the matrix equality

$$\mathcal{J}_{\{e_1,e_2,e_3\}} \circ \mathcal{J}_{\{e_4\}} = \mathcal{J}_{\{e_4\}} \circ \mathcal{J}_{\{e_1,e_2,e_3\}}, \tag{2.5}$$

which leads us to the equations:

$$(e_1) \qquad K_{14}(R_{1224}+R_{1334})+R_{1442}(R_{2114}+R_{2334})+R_{1443}(R_{3114}+R_{3224}) = 0,$$

$$(e_2) \qquad R_{1442}(R_{1224}+R_{1334})+K_{24}(R_{2114}+R_{2334})+R_{2443}(R_{3114}+R_{3224}) = 0,$$

$$(e_3) \qquad R_{1443}(R_{1224}+R_{1334})+R_{2443}(R_{2114}+R_{2334})+K_{34}(R_{3114}+R_{3224}) = 0.$$

We do a cyclic change of indecies $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ in (e_1), (e_2), (e_3), and get the equations

$$(e_1^1) \qquad K_{12}(R_{1332}+R_{1442})+R_{2113}(R_{1223}+R_{1443})+R_{2114}(R_{1224}+R_{1334}) = 0,$$

$$(e_1^2) \qquad R_{2113}(R_{1332}+R_{1442})+K_{13}(R_{1223}+R_{1443})+R_{3114}(R_{1224}+R_{1334}) = 0,$$

$$(e_1^3) \qquad R_{2114}(R_{1332}+R_{1442})+R_{3114}(R_{1223}+R_{1443})+K_{14}(R_{1224}+R_{1334}) = 0.$$

Doing a cyclic change of indecies $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ in (e_1^1), (e_1^2) and (e_3^1), we get

$$(e_2^1) \qquad K_{23}(R_{2113}+R_{2443})+R_{3224}(R_{2114}+R_{2334})+R_{1223}(R_{1332}+R_{1442})=0,$$

$$(e_2^2) \qquad R_{3224}(R_{2113}+R_{2443})+K_{24}(R_{2114}+R_{2334})+R_{1224}(R_{1332}+R_{1442})=0,$$

$$(e_2^3) \qquad R_{1223}(R_{2113}+R_{2443})+R_{1224}(R_{2114}+R_{2334})+K_{12}(R_{1332}+R_{1442})=0.$$

Another cycling change $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ in (e_2^1), (e_2^2) and (e_2^3) will give us

$$(e_3^1) \qquad K_{34}(R_{3114}+R_{3224})+R_{1334}(R_{1223}+R_{1443})+R_{2334}(R_{2113}+R_{2443})=0,$$

$$(e_3^2) \qquad R_{1334}(R_{3114}+R_{3224})+K_{13}(R_{1223}+R_{1443})+R_{1332}(R_{2113}+R_{2443})=0,$$

$$(e_3^3) \qquad R_{2334}(R_{3114}+R_{3224})+R_{1332}(R_{1223}+R_{1443})+K_{23}(R_{2113}+R_{2443})=0.$$

Solving (e_1), (e_2), (e_3) with respect to $R_{1224}+R_{1334}$, $R_{2114}+R_{2334}$, $R_{3114}+R_{3224}$, using *Maple*, we get:

$$R_{1224}+R_{1334}=R_{2114}+R_{2334}=R_{3114}+R_{3224}=0,$$

since the above is in fact the trivial solution to the system of equations (e_1), (e_2), (e_3) which is homogeneous:

$$\begin{pmatrix} K_{12} & R_{1442} & R_{1443} \\ R_{1442} & K_{24} & R_{2443} \\ R_{1443} & R_{2443} & K_{34} \end{pmatrix} \begin{pmatrix} R_{1224}+R_{1334} \\ R_{2114}+R_{2334} \\ R_{3114}+R_{3224} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Analogously, solving the other two homogeneous systems (e_1^1), (e_1^2), (e_1^3) and (e_3^1), (e_3^2), (e_3^3) we get that

$$R_{1332}+R_{1442}=R_{1223}+R_{1443}=R_{1224}+R_{1334}=0,$$

and

$$R_{1223}+R_{1443}=R_{2113}+R_{2443}=R_{3114}+R_{3224}=0.$$

From (2.3) we also get

$$(e_4) \qquad (K_{14}-K_{24})R_{1332}+R_{2113}R_{1443}-R_{2443}R_{1223}=0,$$

$$(e_5) \qquad (K_{14}-K_{34})R_{1223}+R_{2113}R_{1442}-R_{2443}R_{1332}=0,$$

$$(e_6) \qquad (K_{24}-K_{34})R_{2113}+(K_{13}-K_{12})R_{2443}+R_{1223}R_{1442}-R_{1332}R_{1443}=0.$$

By a cycling change of indecies $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ in (e_4), (e_5) and (e_6), we get

$$(e_4^1) \qquad (K_{12} - K_{13})R_{2443} + R_{2114}R_{3224} - R_{3114}R_{2334} = 0,$$

$$(e_4^2) \qquad (K_{12} - K_{14})R_{2334} + R_{3224}R_{2113} - R_{3114}R_{2443} = 0,$$

$$(e_4^3) \qquad (K_{13} - K_{14})R_{3224} + (K_{24} - K_{23})R_{3114} + R_{2334}R_{2113} - R_{2443}R_{2114} = 0.$$

Repeating the same procedure two more times, we get

$$(e_5^1) \qquad (K_{23} - K_{24})R_{3114} + R_{1334}R_{1223} - R_{1224}R_{1443} = 0,$$

$$(e_5^2) \qquad (K_{23} - K_{12})R_{1443} + R_{3224}R_{1334} - R_{1224}R_{3114} = 0,$$

$$(e_5^3) \qquad (K_{24} - K_{12})R_{1334} + (K_{13} - K_{34})R_{1224} + R_{1443}R_{3224} - R_{3114}R_{1223} = 0.$$

and

$$(e_6^1) \qquad (K_{34} - K_{13})R_{1224} + R_{2334}R_{1442} - R_{1332}R_{2114} = 0,$$

$$(e_6^2) \qquad (K_{34} - K_{23})R_{2114} + R_{1334}R_{1442} - R_{1332}R_{1224} = 0,$$

$$(e_6^3) \qquad (K_{13} - K_{23})R_{1442} + (K_{24} - K_{14})R_{1332} + R_{1334}R_{2114} - R_{2334}R_{1224} = 0.$$

Further, from (e_4), (e_5), (e_6); (e_4^1), (e_4^2), (e_4^3); (e_5^1), (e_5^2), (e_5^3); (e_6^1), (e_6^2), (e_6^3) and using that

$$\begin{array}{ll} R_{2113} + R_{2443} = 0, & R_{1332} + R_{1442} = 0 \\ R_{1223} + R_{1443} = 0, & R_{1224} + R_{1334} = 0 \\ R_{2114} + R_{2334} = 0, & R_{3114} + R_{3224} = 0, \end{array}$$

we get the system of equations

$$\left| \begin{array}{l} K_{12} = K_{34} \\ \\ K_{13} = K_{24} \\ \\ K_{14} = K_{23} \end{array} \right. ,$$

with respect to the basis $\{e_1, e_2, e_3, e_4\}$. The latter is equivalent to (M, g) being an Einstein[10]. $\square$

(**c**) $\Longrightarrow$ (**a**) Suppose (M, g) is a four-dimensional Einstein manifold and let $X \in M_p$, $p \in M$ and $X^\perp$ is the orthogonal complement of X. Then $\rho = \lambda\,\mathrm{Id}$, $\lambda = \mathrm{const.}$, and hence

$$\mathcal{J}(X)\circ\mathcal{J}(X^\perp) - \mathcal{J}(X^\perp)\circ\mathcal{J}(X) =$$

$$\mathcal{J}(X)\circ\mathcal{J}(X^\perp) + \mathcal{J}(X)\circ\mathcal{J}(X) - \mathcal{J}(X)\circ\mathcal{J}(X) - \mathcal{J}(X^\perp)\circ\mathcal{J}(X) =$$

$$\mathcal{J}(X)\circ\underbrace{\left[\mathcal{J}(X^\perp) + \mathcal{J}(X)\right]}_{\rho} - \underbrace{\left[\mathcal{J}(X) + \mathcal{J}(X^\perp)\right]}_{\rho}\circ\mathcal{J}(X) =$$

$$\mathcal{J}(X)\circ\rho - \rho\circ\mathcal{J}(X) = \lambda\left(\mathcal{J}(X)\circ\mathrm{Id} - \mathrm{Id}\circ\mathcal{J}(X)\right) = 0.$$

□

Analogously, one can prove, using [9], the following

Corollary 2.1. *Let (M, g) be a three-dimensional Riemannian manifold. Then the next two conditions are equivalent:*

(i) $\mathcal{J}(X)\circ\mathcal{J}(X^\perp) = \mathcal{J}(X^\perp)\circ\mathcal{J}(X)$ *for arbitrary* X, $X^\perp \in M_p$, $p \in M$.

(ii) (M, g) *has a constant sectional curvature* κ *such that* $R(X, Y, Z) = \kappa(g(Y, Z)X - g(X, Z)Y)$, $X, Y, Z \in M_p$.

(**b**) $\Longrightarrow$ (**c**) Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for M_p, $p \in M$. Then the curvature operator matrices $\mathcal{J}_{\{e_1,e_2\}}$ and $\mathcal{J}_{\{e_3,e_4\}}$ have the form:

$$\mathcal{J}_{\{e_1,e_2\}} = \begin{pmatrix} K_{12}+K_{13} & 0 & R_{1223} & R_{1224} \\ 0 & K_{12} & R_{2113} & R_{2114} \\ R_{1223} & R_{2113} & K_{13}+K_{23} & \rho_{34} \\ R_{1224} & R_{2114} & \rho_{34} & K_{14}+K_{24} \end{pmatrix}, \tag{2.6}$$

and

$$\mathcal{J}_{\{e_3,e_4\}} = \begin{pmatrix} K_{13}+K_{14} & \rho_{12} & R_{1443} & R_{1334} \\ \rho_{12} & K_{23}+K_{24} & R_{2443} & R_{2334} \\ R_{1443} & R_{2443} & K_{34} & 0 \\ R_{1334} & R_{2334} & 0 & K_{34} \end{pmatrix}. \tag{2.7}$$

Let also

$$(A) = \begin{pmatrix} K_{12}+K_{13} & 0 & R_{1223} & R_{1224} \\ 0 & K_{12} & R_{2113} & R_{2114} \\ R_{1223} & R_{2113} & K_{13}+K_{23} & \rho_{34} \\ R_{1224} & R_{2114} & \rho_{34} & K_{14}+K_{24} \end{pmatrix}$$
$$\begin{pmatrix} K_{13}+K_{14} & \rho_{12} & R_{1443} & R_{1334} \\ \rho_{12} & K_{23}+K_{24} & R_{2443} & R_{2334} \\ R_{1443} & R_{2443} & K_{34} & 0 \\ R_{1334} & R_{2334} & 0 & K_{34} \end{pmatrix}.$$

Simple computations for the matrix (A) give us:

$$\begin{aligned}
a_{11} &= K_{12}(K_{13}+K_{14}) + R_{1223}R_{1443} + R_{1334}R_{1224},\\
a_{12} &= K_{12}\rho_{12} + R_{1223}R_{2443} + R_{2334}R_{1224},\\
a_{13} &= K_{12}R_{1443} + K_{34}R_{1223},\\
a_{14} &= K_{12}R_{1334} + K_{34}R_{1224},\\
a_{21} &= K_{12}\rho_{12} + R_{1443}R_{2113} + R_{2114}R_{1334},\\
a_{22} &= K_{12}(K_{23}+K_{24}) + R_{2443}R_{2113} + R_{2114}R_{2334},\\
a_{23} &= K_{12}R_{2443} + K_{34}R_{2114},\\
a_{24} &= K_{12}R_{2334} + K_{34}R_{2114},\\
a_{31} &= (K_{13}+K_{13})R_{1223} + (K_{13}+K_{23})R_{1443} + \rho_{12}R_{2113} + \rho_{34}R_{1443},\\
a_{32} &= (K_{23}+K_{24})R_{2113} + (K_{13}+K_{23})R_{2443} + \rho_{12}R_{1223} + \rho_{34}R_{2334},\\
a_{33} &= (K_{13}+K_{23})K_{34} + R_{1223}R_{1443} + R_{2443}R_{2113},\\
a_{34} &= \rho_{34}K_{34} + R_{1223}R_{1334} + R_{2334}R_{2113},\\
a_{41} &= (K_{13}+K_{14})R_{1224} + (K_{14}+K_{24})R_{1334} + \rho_{12}R_{2114} + \rho_{34}R_{1443},\\
a_{42} &= (K_{23}+K_{24})R_{2114} + (R_{14}+K_{24})R_{2334} + \rho_{12}R_{1224} + \rho_{34}R_{2443},\\
a_{43} &= \rho_{34}K_{34} + R_{1443}R_{1224} + R_{2114}R_{2334},\\
a_{44} &= (K_{14}+K_{24})K_{34} + R_{1334}R_{1224} + R_{2114}R_{2334}.
\end{aligned} \tag{2.8}$$

On the other hand let

$$(B) = \begin{pmatrix} K_{13}+K_{14} & \rho_{12} & R_{1443} & R_{1334} \\ \rho_{12} & K_{23}+K_{24} & R_{2443} & R_{2334} \\ R_{1443} & R_{2443} & K_{34} & 0 \\ R_{1334} & R_{2334} & 0 & K_{34} \end{pmatrix} \begin{pmatrix} K_{12} & 0 & R_{1223} & R_{1224} \\ 0 & K_{12} & R_{2113} & R_{2114} \\ R_{1223} & R_{2113} & K_{13}+K_{23} & \rho_{34} \\ R_{1224} & R_{2114} & \rho_{34} & K_{14}+K_{24} \end{pmatrix}.$$

For the matrix (B) elements we derive:

$$\begin{aligned}
&b_{11} = K_{12}(K_{13}+K_{14}) + R_{1223}R_{1443} + R_{1334}R_{1224},\\
&b_{12} = K_{12}\rho_{12} + R_{1443}R_{2113} + R_{2114}R_{1334},\\
&b_{13} = (K_{13}+K_{14})R_{1223} + (K_{13}+K_{23})R_{1443} + \rho_{12}R_{2113} + \rho_{34}R_{1334},\\
&b_{14} = (K_{13}+K_{14})R_{1224} + (K_{14}+K_{24})R_{1334} + \rho_{12}R_{2114} + \rho_{34}R_{1443},\\
&b_{21} = K_{12}\rho_{12} + R_{1223}R_{2443} + R_{2334}R_{1224},\\
&b_{22} = K_{12}(K_{23}+K_{24}) + R_{2443}R_{2113} + R_{2114}R_{2334},\\
&b_{23} = (K_{23}+K_{24})R_{2113} + (K_{13}+K_{23})R_{2443} + \rho_{12}R_{1223} + \rho_{34}R_{2334},\\
&b_{24} = (K_{23}+K_{24})R_{2114} + (K_{14}+K_{24})R_{2334} + \rho_{12}R_{1224} + \rho_{34}R_{2443},\\
(2.9)\quad &b_{31} = K_{12}R_{1443} + K_{34}R_{1223},\\
&b_{32} = K_{12}R_{2443} + K_{34}R_{2113},\\
&b_{33} = (K_{13}+K_{23})K_{34} + R_{1223}R_{1443} + R_{2443}R_{2113},\\
&b_{34} = \rho_{34}K_{34} + R_{1443}R_{1224} + R_{2114}R_{2443},\\
&b_{41} = K_{12}R_{1334} + K_{34}R_{1224},\\
&b_{42} = K_{12}R_{2334} + K_{34}R_{2114},\\
&b_{43} = \rho_{34}K_{34} + R_{1223}R_{1334} + R_{2334}R_{2113},\\
&b_{44} = (K_{14}+K_{24})K_{34} + R_{1334}R_{1224} + R_{2114}R_{2334}.
\end{aligned}$$

From **[C2]** we have $a_{12} = b_{12}$ and according to (2.6) and (2.7), we get:

$$(e_7)\qquad R_{1223}R_{2443} + R_{2334}R_{1224} - R_{1443}R_{2113} - R_{2114}R_{1334} = 0,$$

$$(e_8)\qquad R_{1223}R_{1334} + R_{2334}R_{2113} - R_{1443}R_{1224} - R_{2114}R_{2443} = 0.$$

By a cycling change of indecies $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ in (e_7) and (e_8), we get

$$(e_7^1) \qquad R_{2334}R_{3114} + R_{1443}R_{1332} - R_{2114}R_{3224} - R_{1223}R_{1442} = 0,$$

$$(e_8^1) \qquad R_{2334}R_{1442} + R_{1443}R_{3224} - R_{2114}R_{1332} - R_{1223}R_{3114} = 0.$$

We solve, using *Maple*, $(e_7), (e_8), (e_7^1)$ and (e_8^1) together and arrive at the homogeneous system

$$(2.10) \qquad \begin{pmatrix} R_{2443} & R_{1224} & -R_{2113} & -R_{1334} \\ R_{1334} & R_{2113} & -R_{1224} & -R_{2443} \\ -R_{1442} & R_{3114} & R_{1332} & -R_{3224} \\ -R_{3114} & R_{1442} & R_{3224} & -R_{1332} \end{pmatrix} \begin{pmatrix} R_{1223} \\ R_{2334} \\ R_{1443} \\ R_{2114} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving (2.8) with respect to $R_{1223}, R_{2334}, R_{1443}$ and R_{2114}, we get at least the trivial solution:

$$(2.11) \qquad R_{2114} = R_{1223} = R_{1443} = R_{2334} = 0.$$

We also have $\mathcal{J}_{\{e_1,e_3\}} = \mathcal{J}_{\{e_2,e_4\}}$ and $\mathcal{J}_{\{e_1,e_4\}} = \mathcal{J}_{\{e_2,e_3\}}$, and using (2.9), it follows that

$$(2.12) \qquad \left| \begin{array}{l} R_{2113}(R_{1332} + R_{1442}) + R_{1334}(R_{3114} + R_{3224}) = 0 \\ R_{1334}(K_{12} - K_{14} - K_{24}) + R_{1224}(K_{34} - K_{13} - K_{14}) = 0 \\ R_{2443}(K_{12} - K_{13} - K_{23}) + R_{2113}(K_{34} - K_{23} - K_{24}) = 0 \\ R_{1224}(R_{1332} + R_{1442}) + R_{2443}(R_{3114} + R_{3224}) = 0 \\ R_{3224}(R_{2113} + R_{2443}) + R_{1442}(R_{1224} + R_{1334}) = 0 \\ R_{1442}(K_{23} - K_{12} - K_{13}) + R_{1332}(K_{14} - K_{34} - K_{13}) = 0 \\ R_{1332}(R_{2113} + R_{2443}) + R_{3114}(R_{1224} + R_{1334}) = 0 \end{array} \right. .$$

We solve (2.10) with respect to the tensor R components R_{1332}, R_{1442}, R_{3114}, R_{3224}, R_{1224}, R_{1334}, R_{2113}, R_{2114} using, for example *Maple*, and as a result we get

$$(2.13) \qquad \begin{array}{l} R_{1224}(-K_{34} + K_{13} + K_{14}) = R_{1334}(K_{12} - K_{14} - K_{24}) \\ R_{2113}(-K_{34} + K_{23} + K_{24}) = R_{2443}(K_{12} - K_{13} - K_{23}) \end{array}$$

and

$$(2.14) \qquad R_{1332} = R_{3224} = R_{1442} = R_{3114} = 0.$$

Further, by changing the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with the orthonormal basis $\left\{ \dfrac{e_1 - e_2}{\sqrt{2}}, \dfrac{e_1 + e_2}{\sqrt{2}}, \dfrac{e_3 - e_4}{\sqrt{2}}, \dfrac{e_3 + e_4}{\sqrt{2}} \right\}$ and using (2.11) and

(2.12) we receive a new system of equations with respect to the tensor curvature components which is equivalent to (2.10). From there we can conclude that

$$R_{1334} = R_{2443} = 0. \tag{2.15}$$

From (2.9), (2.11) and (2.12) it follows that all of the components R_{ijjk} are equal to zero for all $i, j, k = 1, 2, 3, 4$.

If we reformulate the second and third equation in (2.10) by changing the basis as shown above and transforming the curvature components using (2.9), (2.11) and (2.12), we get

$$\begin{array}{l} (K_{13} - K_{23} + K_{14} - K_{24})(R_{1432} + R_{1342} + K_{12}) = 0 \\ (K_{13} + K_{23} - K_{14} - K_{24})(R_{1432} + R_{1342} + K_{34}) = 0 \end{array}. \tag{2.16}$$

By a cycling change of indecies $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$ above and some extra tedious computations, we get the system:

$$\left| \begin{array}{l} (K_{13} + K_{14} - K_{23} - K_{24})(K_{12} + R_{1342} - R_{1423}) = 0 \\ (K_{12} + K_{14} - K_{23} - K_{34})(K_{13} + R_{1234} - R_{1423}) = 0 \\ (K_{12} + K_{13} - K_{24} - K_{34})(K_{14} + R_{1234} - R_{1342}) = 0 \\ (K_{12} + K_{24} - K_{13} - K_{34})(K_{23} + R_{1234} - R_{1342}) = 0 \\ (K_{12} + K_{23} - K_{14} - K_{34})(K_{24} + R_{1234} - R_{1423}) = 0 \\ (K_{13} + K_{23} - K_{14} - K_{24})(K_{34} + R_{1342} - R_{1432}) = 0 \end{array} \right. . \tag{2.17}$$

Solving (2.14) and (2.15) with respect to the sectional curvature components K_{12}, K_{13} and K_{14}, it follows that

$$K_{14} = K_{23}, \quad K_{13} = K_{24}, \quad K_{12} = K_{34}, \tag{2.18}$$

and since the basis $\{e_1, e_2, e_3, e_4\}$ has been arbitrary chosen in M_p, it follows that (M, g) is Einstein [10]. □

(**b**) $\Longrightarrow$ (**a**) If (M, g) is an Einstein manifold then $\rho = \lambda \,\mathrm{Id}$, $\lambda =$ const., and if α is a 2-plane in M_p, $p \in M$, it follows that

$$\begin{gathered} \mathcal{J}(\alpha) \circ \mathcal{J}(\alpha^\perp) - \mathcal{J}(\alpha^\perp) \circ \mathcal{J}(\alpha) = \\ \mathcal{J}(\alpha) \circ \mathcal{J}(\alpha^\perp) + \mathcal{J}(\alpha) \circ \mathcal{J}(\alpha) - \mathcal{J}(\alpha) \circ \mathcal{J}(\alpha) - \mathcal{J}(\alpha^\perp) \circ \mathcal{J}(\alpha) = \\ \mathcal{J}(\alpha) \circ \underbrace{\left[\mathcal{J}(\alpha^\perp) + \mathcal{J}(\alpha)\right]}_{\rho} - \underbrace{\left[\mathcal{J}(\alpha) + \mathcal{J}(\alpha^\perp)\right]}_{\rho} \circ \mathcal{J}(\alpha) = \\ \mathcal{J}(\alpha) \circ \rho - \rho \circ \mathcal{J}(\alpha) = \lambda \left(\mathcal{J}(\alpha) \circ \mathrm{Id} - \mathrm{Id} \circ \mathcal{J}(\alpha)\right) = 0. \end{gathered}$$

That completes the proof. □

3. New approaches and results

Recently Gilkey, Puffini and Videv [4] were able to generalize the results above. They define $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$ to be a 0-model if $\langle\cdot,\cdot\rangle$ is a non-degenerate inner product of signature (p, q) on a finite dimensional vector space V of dimension $m = p + q$ and if $A \in \otimes^4 V^*$ is an algebraic curvature tensor.

Let $\mathrm{Gr}_{r,s}(V, \langle\cdot,\cdot\rangle)$ be the Grassmannian of all non-degenerate linear subspaces of V which have signature (r, s); the pair (r, s) is said to be *admissible* if and only if $Gr_{r,s}(V, \langle\cdot,\cdot\rangle)$ is non-empty and does not consist of a single point or, equivalently, if the inequalities $0 \le r \le p$, $0 \le s \le q$, and $1 \le r + s \le m - 1$ are satisfied. Let $[A, B] := AB - BA$ denote the commutator of two linear maps. Then they establish the following result:

Theorem 3.1. *Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, A)$ be a 0-model. The following assertions are equivalent; if any is satisfied, then we shall say that $\mathfrak{M}$ is a* Puffini–Videv *0-model.*

1. *There exists (r_0, s_0) admissible so that $\mathcal{J}(\pi) \circ \mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp) \circ \mathcal{J}(\pi)$ for all $\pi \in Gr_{r_0,s_0}(V, \langle\cdot,\cdot\rangle)$.*
2. *$\mathcal{J}(\pi) \circ \mathcal{J}(\pi^\perp) = \mathcal{J}(\pi^\perp) \circ \mathcal{J}(\pi)$ for every non-degenerate subspace π.*
3. *$[\mathcal{J}(\pi), \rho] = 0$ for every non-degenerate subspace π.*

We say that $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, A)$ is *decomposible* if there exists a non-trivial orthogonal decomposition $V = V_1 \oplus V_2$ which decomposes $A = A_1 \oplus A_2$; in this setting, we shall write $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$ where $\mathfrak{M}_i := (V_i, \langle\cdot,\cdot\rangle|_{V_i}, A_i)$. One says that $\mathfrak{M}$ is *indecomposible* if $\mathfrak{M}$ is not decomposible.

By Theorem 3.1, any Einstein 0-model is Puffini–Videv. More generally, the direct sum of Einstein Puffini–Videv models is again Puffini–Videv; the converse holds in the Riemannian setting:

Theorem 3.2. *Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, A)$ be a Riemannian 0-model. Then $\mathfrak{M}$ is Puffini–Videv if and only if $\mathfrak{M} = \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_k$ where the $\mathfrak{M}_i$ are Einstein.*

In the pseudo-Riemannian setting, a somewhat weaker result can be established. One says that a 0-model is *pseudo-Einstein* either if the Ricci operator ρ has only one real eigenvalue λ or if the Ricci operator ρ has two complex eigenvalues λ_1, λ_2 with $\bar{\lambda}_1 = \lambda_2$. This does not imply that ρ is diagonalizable in the higher signature setting and hence $\mathfrak{M}$ need not be Einstein.

Theorem 3.3. *Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, A)$ be a 0-model of arbitrary signature. If $\mathfrak{M}$ is Puffini–Videv, then we may decompose $\mathfrak{M} = \mathfrak{M}_1 \oplus \cdots \oplus \mathfrak{M}_k$ as the direct sum of pseudo-Einstein 0-models $\mathfrak{M}_i$.*

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ЧЕТИРИМЕРНИ РИМАНОВИ МНОГООБРАЗИЯ С КОМУТИРАЩИ ОПЕРАТОРИ НА ЯКОБИ

Живко Желев, Мария Иванова и Веселин Видев

Резюме. Разглеждат се четиримерни риманови многообразия с комутиращи оператори на Якоби върху двумерни площадки и техните ортогонални подпространства.

По-точно разглежда се операторът на Якоби $\mathcal{J}(X)$, който комутира с оператора $\mathcal{J}(X^{\perp})$, индуциран от ортогоналното допълнение $X^{\perp}$, т. е. $\mathcal{J}(X) \circ \mathcal{J}(X^{\perp}) = \mathcal{J}(X^{\perp}) \circ \mathcal{J}(X)$.

Накрая са цитирани и някои нови резултати, получени от П. Гилки, Е. Пуфини и В. Видев.