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ON THE MOVING BOUNDARY HITTING PROBABILITY FOR A BROWNIAN MOTION

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Abstract. We consider the probability that a Brownian motion hits a moving two-sided boundary by a certain moment. In some special cases we find formulae for this probability.

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1. Introduction

Let $(B_s)_{s \geq t}$ be a Brownian motion with unit volatility and no drift, $B_t = x$. Let T be an arbitrary fixed time-horizon, $T \geq t$, and $g(s) < f(s)$ be two smooth real functions, defined at least for $s \in [t;T]$, such that $g(t) \leq x \leq f(t)$. Consider the hitting time $\tau = \inf\{s \in [t;T] \mid B_s = f(s) \text{ or } B_s = g(s)\}$, where $\inf \varnothing = T$. We define the functions

$$v_f(t,x) = P_{t,\,x}\left(B_\tau = f\left(\tau\right)\right),$$

$$v_g(t,x) = P_{t,\,x}\left(B_\tau = g\left(\tau\right)\right).$$

In 1960 T. W. Anderson [1] discovered the crossing probabilities for rectilinear boundaries with no horizon — two straight lines that are parallel or cross on the left of the origin. In 1964 A. V. Skorokhod [2] found the probability of going out of the domain through a little “door” at the horizon; his formula holds for rectilinear boundaries. In 1967 L. A. Shepp [3] found a formula for the expectation of the first hitting time for a two-sided symmetric square-root boundary with no horizon. In 1971 A. A. Novikov [4] solved the same problem for a one-sided square-root boundary. In 1981 he published a formula [5] for the probability of going out of the domain through the horizon; it holds for curvilinear boundaries that are close to each other. A little later in the same year A. V. Mel'nikov and D. I. Hadžiev [6] published a solution to a similar problem for Gaussian martingales. In 1999 A. Novikov, V. Frishling and N. Kordzakhia [7] found approximate formulae for the crossing probabilities both for a one-sided and a two-sided boundary with a horizon; they were able to derive exact formulae for a one-sided and a two-sided symmetric square-root boundary.

In this paper we find formulae for $v_f(t,x)$ and $v_g(t,x)$ for parallel rectilinear boundaries and arbitrary square-root boundaries.

2. Calculation of $v_f(t,x)$

According to [8], the function $v_f(t,x)$ is a solution to the problem

$$(1)\qquad \left|\begin{array}{ll} \dfrac{\partial v_f}{\partial t} + \dfrac{1}{2}\cdot\dfrac{\partial^2 v_f}{\partial x^2} = 0, & t < T,\ x \in (g(t); f(t)) \\ v_f(T,x) = 0, & x \in (g(T); f(T)) \\ v_f(t,g(t)) = 0, & t \le T \\ v_f(t,f(t)) = 1, & t \le T. \end{array}\right.$$

The equation is simple enough, but the boundary is too complicated. To get a rectangular boundary, set

$$h(t) = f(t) - g(t) > 0, \quad v_f(t,x) = v_1\left(t,\ \frac{x-g(t)}{h(t)}\right).$$

Then the function $v_1(t,x)$ is a solution to the problem

$$(2)\qquad \left|\begin{array}{ll} \dfrac{\partial v_1}{\partial t} - \dfrac{h'(t)\,.\,x + g'(t)}{h(t)}\cdot\dfrac{\partial v_1}{\partial x} + \dfrac{1}{2}\cdot\dfrac{1}{h^2(t)}\cdot\dfrac{\partial^2 v_1}{\partial x^2} = 0 & \\ v_1(T,x) = 0, \quad 0 < x < 1 & \\ v_1(t,0) = 0, \quad t \le T & \\ v_1(t,1) = 1, \quad t \le T. & \end{array}\right.$$

2.1. Two "parallel" curves: $h(t) = c = const., \forall t \le T\ (c > 0)$. Therefore $h'(t) = 0$ and the equation takes the form

$$\frac{\partial v_1}{\partial t} - \frac{g'(t)}{c}\cdot\frac{\partial v_1}{\partial x} + \frac{1}{2c^2}\cdot\frac{\partial^2 v_1}{\partial x^2} = 0.$$

2.1.1. *Two parallel straight lines*: $g(t) = bt + c_1$, $\forall t \leq T$. Then $f(t) = bt + c_2$, $c_2 > c_1$, $c = c_2 - c_1 > 0$, $g'(t) = b$, and the equation becomes

$$\frac{\partial v_1}{\partial t} - \frac{b}{c} \cdot \frac{\partial v_1}{\partial x} + \frac{1}{2c^2} \cdot \frac{\partial^2 v_1}{\partial x^2} = 0.$$

Let $\kappa = \dfrac{1}{2c^2} > 0$, $\lambda = \dfrac{b}{c}$. Then we have to solve the problem

$$\left|\begin{array}{l} \dfrac{\partial v_1}{\partial t} - \lambda \cdot \dfrac{\partial v_1}{\partial x} + \kappa \cdot \dfrac{\partial^2 v_1}{\partial x^2} = 0 \\ v_1(T, x) = 0, \quad \forall x \in (0; 1) \\ v_1(t, 0) = 0, \quad \forall t \in (-\infty; T] \\ v_1(t, 1) = 1, \quad \forall t \in (-\infty; T]. \end{array}\right.$$

We would prefer an initial condition, so we set

$$v_2(t, x) = v_1(T - t, x)$$

and reformulate the problem as follows:

$$\left|\begin{array}{l} -\dfrac{\partial v_2}{\partial t} - \lambda \cdot \dfrac{\partial v_2}{\partial x} + \kappa \cdot \dfrac{\partial^2 v_2}{\partial x^2} = 0 \\ v_2(0, x) = 0, \quad \forall x \in (0; 1) \\ v_2(t, 0) = 0, \quad \forall t \in [0; +\infty) \\ v_2(t, 1) = 1, \quad \forall t \in [0; +\infty). \end{array}\right.$$

After the Laplace transformation $V(p, x) = L[v_2(t, x)]$ we get the problem

$$\text{(3)} \qquad \left|\begin{array}{l} \kappa \,.\, V'' - \lambda \,.\, V' - p \,.\, V = 0 \\ V(0) = 0 \\ V(1) = \dfrac{1}{p} \end{array}\right.$$

(V is considered a function of x, and p is just a parameter).

The corresponding characteristic equation is $\kappa\nu^2 - \lambda\nu - p = 0$ with $D = \lambda^2 + 4\kappa p > 0$, because $p > 0$, so $\nu_{1,2} = \dfrac{\lambda \pm \sqrt{\lambda^2 + 4\kappa p}}{2\kappa}$ and

$$V(x) = \left(C_1 \,.\, \cosh \frac{\sqrt{\lambda^2 + 4\kappa p} \,.\, x}{2\kappa} + C_2 \,.\, \sinh \frac{\sqrt{\lambda^2 + 4\kappa p} \,.\, x}{2\kappa} \right) \exp\left(\frac{\lambda x}{2\kappa}\right).$$

From $V(0)=0$ and $V(1)=\dfrac{1}{p}$ we find $C_1=0$, $C_2=\dfrac{\exp\left(-\frac{\lambda}{2\kappa}\right)}{p\ .\ \sinh\frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}}$. So

$$V(x)=\frac{\sinh\frac{\sqrt{\lambda^2+4\kappa p}\ .\ x}{2\kappa}}{p\ .\ \sinh\frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}}\ \exp\left(\frac{\lambda(x-1)}{2\kappa}\right).$$

Thus we have just proved the following theorem (where L^{-1} stands for the reversed Laplace transformation):

Theorem 1. *If* $g(t)=bt+c_1$, $f(t)=bt+c_2$, $\forall t\le T$, *and* $c_2>c_1$, *then* $v_f(t,x)=v_1\left(t,\dfrac{x-bt-c_1}{c}\right)$, $v_1(t,x)=v_2(T-t,x)$, $v_2(t,x)=L^{-1}\left[V(p,x)\right]$, $V(p,x)=\dfrac{\sinh\frac{\sqrt{\lambda^2+4\kappa p}\ .\ x}{2\kappa}}{p\ .\ \sinh\frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}}\ \exp\left(\dfrac{\lambda(x-1)}{2\kappa}\right)$, $\kappa=\dfrac{1}{2c^2}$, $\lambda=\dfrac{b}{c}$, $c=c_2-c_1$.

Fortunately, the function $V(p,x)$ can be explicitly transformed to the function $v_2(t,x)$ and then back to $v_f(t,x)$. According to [9], we have

$$v_2(t,x)=L^{-1}\left[V(p,x)\right]=\sum_{p_n}\ \operatorname*{res}_{p_n}\ V(p,x)\exp(pt)=$$

$$=\left(\frac{\sinh\frac{\lambda x}{2\kappa}}{\sinh\frac{\lambda}{2\kappa}}+2\kappa\pi\sum_{n=1}^{\infty}\frac{(-1)^n.\,n.\ \sin(n\pi x).\exp\left\{-\left(\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}\right)t\right\}}{\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}}\right)\exp\left(\frac{\lambda(x-1)}{2\kappa}\right).$$

The single addend is the residuum at $p_0=0$. Its value must be considered equal to x when $\lambda=0$. The n-th term of the infinite sum is the residuum at $p_n=-\kappa n^2\pi^2-\frac{\lambda^2}{4\kappa}$, $n\in\mathbb{N}$. Then

$$v_1(t,x)=v_2(T-t,x)=$$

$$=\left(\frac{\sinh\frac{\lambda x}{2\kappa}}{\sinh\frac{\lambda}{2\kappa}}+2\kappa\pi\sum_{n=1}^{\infty}\frac{(-1)^n.n.\sin(n\pi x).\exp\left\{\left(\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}\right)(t-T)\right\}}{\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}}\right)\exp\left(\frac{\lambda(x-1)}{2\kappa}\right).$$

Substituting κ and λ in the last expression, we get

$$\left(\frac{\sinh(bcx)}{\sinh(bc)}+2\pi\sum_{n=1}^{\infty}\frac{(-1)^n.\,n.\ \sin(n\pi x).\exp\left\{\left(n^2\pi^2+b^2c^2\right)\frac{t-T}{2c^2}\right\}}{n^2\pi^2+b^2c^2}\right)\exp\left\{bc(x-1)\right\}.$$

Finally, the probability we are looking for is equal to

$$v_f(t,x) = v_1\left(t,\ \frac{x-bt-c_1}{c}\right)$$

and can be found from the expression above.

Theorem 2. *If* $g(t) = bt + c_1$, $f(t) = bt + c_2$, $\forall t \le T$, *and* $c_2 > c_1$, *then*

$$v_f(t,x) = e^{b(x-bt-c_2)} \cdot \left(\frac{\sinh\left\{b(x-bt-c_1)\right\}}{\sinh(bc)} + \right.$$

$$\left. + \, 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n \cdot \sin\left(n\pi \frac{x-bt-c_1}{c}\right) \cdot \exp\left\{\left(n^2\pi^2 + b^2c^2\right)\frac{t-T}{2c^2}\right\}}{n^2\pi^2 + b^2c^2} \right),$$

where the single addend is equal to $\frac{x-c_1}{c}$ *when* $b = 0$, *and* $c = c_2 - c_1$.

To check, let $T \to +\infty$; we may do this, because the series is convergent uniformly with respect to $T \in [t; +\infty)$; thus we obtain a formula for the case when there is no horizon.

Corollary 1. *If* $g(t) = bt + c_1$, $f(t) = bt + c_2$, $\forall t \in \mathbb{R}$, $c = c_2 - c_1 > 0$ *and there is no horizon, then*

$$v_f(t,x) = \begin{cases} \dfrac{e^{2b(x-bt-c_1)} - 1}{e^{2bc} - 1} & \text{for } b \ne 0; \\[2ex] \dfrac{x - c_1}{c} & \text{for } b = 0. \end{cases}$$

The formula for $b = 0$ is well-known from the martingale theory. The formula for $b \ne 0$ can be found (in different denotation) in [1] as Theorem 4.1 on page 175.

Again, let $c_1 \to -\infty$ in Corollary 1; thus we get the solution for a one-sided boundary with no horizon.

Corollary 2. *If* $f(t) = bt + c_2$, $\forall t \in \mathbb{R}$, *and there are no lower boundary and no horizon, then*

$$v_f(t,x) = \begin{cases} e^{2b(x-bt-c_2)} & \text{for } b > 0; \\ 1 & \text{for } b \le 0. \end{cases}$$

Unfortunately, this technique cannot be applied to Theorem 2: the series is not uniformly convergent with respect to $c_1 \in (-\infty; x_0], \forall x_0 \le x - bt$.

Corollary 2 can be verified as well by means of Kendall's famous formula.

2.2. Square-root boundaries. If $g(t) = a\sqrt{t+\gamma} + c_0$, $f(t) = b\sqrt{t+\gamma} + c_0$, $\forall t \in (-\gamma; T]$, $b > a$, $T+\gamma > 0$, then the problem (2) takes the form

$$\left|\begin{array}{ll} \dfrac{\partial v_1}{\partial t} - \dfrac{1}{2(t+\gamma)} \cdot \left(x + \dfrac{a}{b-a}\right) \cdot \dfrac{\partial v_1}{\partial x} + \dfrac{1}{2} \cdot \dfrac{1}{(b-a)^2(t+\gamma)} \cdot \dfrac{\partial^2 v_1}{\partial x^2} = 0 & \\ v_1(T,x) = 0, \quad \forall x \in (0;1) & \\ v_1(t,0) = 0, \quad \forall t \in (-\gamma\,; T] & \\ v_1(t,1) = 1, \quad \forall t \in (-\gamma\,; T]. & \end{array}\right.$$

It is essential that the multiplier $(t+\gamma)$ is raised to the same power in both denominators; this happens for square-root boundaries only. That is why, by multiplying the equation by $2(t+\gamma)$ we can ensure that the variable t takes part in only one coefficient. Thus we obtain

$$2(t+\gamma) \cdot \frac{\partial v_1}{\partial t} - \left(x + \frac{a}{b-a}\right) \cdot \frac{\partial v_1}{\partial x} + \frac{1}{(b-a)^2} \cdot \frac{\partial^2 v_1}{\partial x^2} = 0.$$

We can get rid of the multiplier $(t+\gamma)$ by means of a suitable substitution. Let $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma), x\right)$; then the function $v_2(t,x)$ is a solution to the problem

$$\left|\begin{array}{ll} \dfrac{\partial v_2}{\partial t} - \left(x + \dfrac{a}{b-a}\right) \cdot \dfrac{\partial v_2}{\partial x} + \dfrac{1}{(b-a)^2} \cdot \dfrac{\partial^2 v_2}{\partial x^2} = 0 & \\ v_2\left(\frac{1}{2}\ln(T+\gamma), x\right) = 0, & \forall x \in (0;1) \\ v_2(t,0) = 0, & \forall t \in \left(-\infty; \frac{1}{2}\ln(T+\gamma)\right] \\ v_2(t,1) = 1, & \forall t \in \left(-\infty; \frac{1}{2}\ln(T+\gamma)\right]. \end{array}\right.$$

Finally, let $v_2(t,x) = v_3\left(-t + \frac{1}{2}\ln(T+\gamma), x\right)$ in order to obtain an initial condition instead of the final one. Therefore we have

$$\left|\begin{array}{ll} -\dfrac{\partial v_3}{\partial t} - \left(x + \dfrac{a}{b-a}\right) \cdot \dfrac{\partial v_3}{\partial x} + \dfrac{1}{(b-a)^2} \cdot \dfrac{\partial^2 v_3}{\partial x^2} = 0 & \\ v_3(0,x) = 0, & \forall x \in (0;1) \\ v_3(t,0) = 0, & \forall t \in [0; +\infty) \\ v_3(t,1) = 1, & \forall t \in [0; +\infty). \end{array}\right.$$

Apply the Laplace transformation: $V(p,x) = L[v_3(t,x)]$. Then

$$\left| \begin{array}{l} \dfrac{1}{(b-a)^2} \cdot V'' - \left(x + \dfrac{a}{b-a} \right) \cdot V' - p \cdot V = 0 \\ V(0) = 0 \\ V(1) = \dfrac{1}{p} \end{array} \right.$$

(V is considered a function of x, and p is just a parameter).

This boundary-value problem has a unique solution; the solution is an analytical function: $V(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. The differential equation turns into the following equation for the coefficients of the series:

$$\frac{1}{(b-a)^2}(n+2)(n+1)c_{n+2} - \left(nc_n + \frac{a}{b-a}(n+1)c_{n+1} \right) - pc_n = 0, \ n \in \mathbb{N}_0;$$

$$\text{i.e.} \qquad c_{n+2} = \frac{a(b-a)}{n+2}\, c_{n+1} + \frac{(b-a)^2(n+p)}{(n+2)(n+1)}\, c_n\ , \quad n \in \mathbb{N}_0\, .$$

We have to find c_0 and c_1 in order to specify the sequence $(c_n)_{n=0}^{\infty}$. From $V(0) = 0$ it follows that $c_0 = 0$. Let $c_1 = c,\ c_n = c\alpha_n\ ,\ n \in \mathbb{N}_0$. Then

$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \alpha_{n+2} = \dfrac{a(b-a)}{n+2}\, \alpha_{n+1} + \dfrac{(b-a)^2(n+p)}{(n+2)(n+1)}\, \alpha_n\ , \ \forall n \in \mathbb{N}_0;$

$V(x) = c \,.\, \sum\limits_{n=0}^{\infty} \alpha_n x^n = c \,.\, \sum\limits_{n=1}^{\infty} \alpha_n x^n$; from the boundary condition $V(1) = \dfrac{1}{p}$ it follows that $c = \dfrac{1}{p \,.\, \sum\limits_{n=1}^{\infty} \alpha_n}$. We have just proved the following

Theorem 3. *If* $g(t) = a\sqrt{t+\gamma} + c_0$, $f(t) = b\sqrt{t+\gamma} + c_0$, $\forall t \in (-\gamma; T]$, $b > a$, $T + \gamma > 0$, *then* $v_f(t,x) = v_1\left(t, \frac{x - c_0 - a\sqrt{t+\gamma}}{(b-a)\sqrt{t+\gamma}}\right)$, $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma),\, x\right)$, *where* $v_2(t,x) = v_3\left(-t + \frac{1}{2}\ln(T+\gamma),\, x\right)$ *and* $v_3(t,x) = L^{-1}\left[V(p,x)\right]$;

$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \alpha_{n+2} = \dfrac{a(b-a)}{n+2}\, \alpha_{n+1} + \dfrac{(b-a)^2(n+p)}{(n+2)(n+1)}\, \alpha_n\ , \ \forall n \in \mathbb{N}_0\,;$

$V(p,x) = c \,.\, \sum\limits_{n=1}^{\infty} \alpha_n x^n, \ c = \dfrac{1}{p \,.\, \sum\limits_{n=1}^{\infty} \alpha_n}\,.$

When $a = 0$, we can obtain an explicit formula for $(\alpha_n)_{n=0}^{\infty}$. Indeed, $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_{n+2} = \dfrac{b^2(n+p)}{(n+2)(n+1)}\,\alpha_n$, $\forall n \in \mathbb{N}_0$; so $\alpha_{2m} = 0, \forall m \in \mathbb{N}_0$. Let $\beta_m = \alpha_{2m+1}$, $\forall m \in \mathbb{N}_0$; then $\beta_0 = 1$, $\beta_{m+1} = \dfrac{b^2(2m+1+p)}{(2m+3)(2m+2)}\,\beta_m$, i.e. $\beta_m = \dfrac{b^2(2m-1+p)}{(2m+1)(2m)}\,\beta_{m-1}$, hence $\beta_m = \frac{b^{2m}}{(2m+1)!}\prod\limits_{\widetilde{m}=1}^{m}(2\widetilde{m}-1+p)$.

Corollary 3. *If* $g(t) = c_0$ *and* $f(t) = b\sqrt{t+\gamma} + c_0$, $\forall t \in (-\gamma; T]$, $b > 0$, $T+\gamma > 0$, *then* $v_f(t,x) = v_1\left(t,\ \dfrac{x - c_0}{b\sqrt{t+\gamma}}\right)$, *where* $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma), x\right)$, $v_2(t,x) = v_3\left(-t + \frac{1}{2}\ln(T+\gamma)\,x\right)$, $v_3(t,x) = L^{-1}\left[V(p,x)\right]$; $c = \dfrac{1}{p\,.\sum\limits_{m=0}^{\infty}\beta_m}$,

$$\beta_0 = 1,\ \beta_m = \frac{b^{2m}}{(2m+1)!}\prod_{\widetilde{m}=1}^{m}(2\widetilde{m}-1+p)\,,\ \forall m \in \mathbb{N};\ V(p,x) = c\,.\sum_{m=0}^{\infty}\beta_m x^{2m+1}.$$

3. Calculation of $v_g(t,x)$

This function satisfies requirements similar to (1) save that $v_g(t, f(t)) = 0$, $v_g(t, g(t)) = 1$. This change propagates through the calculations.

3.1. Two parallel straight lines. Now (3) changes in this way:

$$\left|\begin{array}{l} \kappa\,.\,V'' - \lambda\,.\,V' - p\,.\,V = 0 \\ V(0) = \dfrac{1}{p} \\ V(1) = 0. \end{array}\right.$$

After the substitution $V(x) = W(1-x)$ we get the problem

$$\left|\begin{array}{l} \kappa\,.\,W'' + \lambda\,.\,W' - p\,.\,W = 0 \\ W(0) = 0 \\ W(1) = \dfrac{1}{p}\ . \end{array}\right.$$

This is the same problem as (3), only λ has changed its sign. Then

$$W(x) = \frac{\sinh\frac{\sqrt{\lambda^2+4\kappa p}\ .\ x}{2\kappa}}{p\,.\,\sinh\frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}}\ \exp\left(\frac{\lambda(1-x)}{2\kappa}\right),$$

$$V(x) = \frac{\sinh \frac{\sqrt{\lambda^2+4\kappa p}\,.\,(1-x)}{2\kappa}}{p\,.\,\sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}} \ \exp\left(\frac{\lambda x}{2\kappa}\right),$$

which proves the next statement.

Theorem 4. *If* $g(t) = bt + c_1$, $f(t) = bt + c_2$, $\forall t \le T$, *and* $c_2 > c_1$, *then* $v_g(t,x) = v_1\left(t, \dfrac{x - bt - c_1}{c}\right)$, $v_1(t,x) = v_2(T-t,x)$, $v_2(t,x) = L^{-1}\left[V(p,x)\right]$,
$V(p,x) = \dfrac{\sinh \frac{\sqrt{\lambda^2+4\kappa p}\,.\,(1-x)}{2\kappa}}{p\,.\,\sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}} \ \exp\left(\dfrac{\lambda x}{2\kappa}\right)$, $\kappa = \dfrac{1}{2c^2}$, $\lambda = \dfrac{b}{c}$, $c = c_2 - c_1$.

Theorem 5 can be deduced from Theorem 4 as Theorem 2 was deduced from Theorem 1. Or we may notice that the changes in the formulae are equivalent to swapping the lower and the upper boundary: we replace $x - g(t)$ with $f(t) - x$ and vice versa as well as b with $-b$ in Theorem 2.

Theorem 5. *If* $g(t) = bt + c_1$, $f(t) = bt + c_2$, $\forall t \le T$, *and* $c_2 > c_1$, *then*

$$v_g(t,x) = e^{b(x-bt-c_1)} \,.\left(\frac{\sinh\left\{b(bt+c_2-x)\right\}}{\sinh(bc)} + \right.$$
$$\left. + \ 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n\,.\,n\,.\,\sin\left(n\pi\frac{bt+c_2-x}{c}\right).\exp\left\{\left(n^2\pi^2+b^2c^2\right)\frac{t-T}{2c^2}\right\}}{n^2\pi^2+b^2c^2} \right),$$

where the single addend is equal to $\frac{c_2-x}{c}$ *when* $b = 0$, *and* $c = c_2 - c_1$.

Corollary 4. *If* $g(t) = bt + c_1$, $f(t) = bt + c_2$, $\forall t \in \mathbb{R}$, $c = c_2 - c_1 > 0$ *and there is no horizon, then*

$$v_g(t,x) = \begin{cases} \dfrac{e^{-2b(bt+c_2-x)} - 1}{e^{-2bc} - 1} & \text{for } b \neq 0; \\ \dfrac{c_2 - x}{c} & \text{for } b = 0. \end{cases}$$

Corollary 5. *If* $g(t) = bt + c_1$, $\forall t \in \mathbb{R}$, *and there are no upper boundary and no horizon, then*

$$v_g(t,x) = \begin{cases} e^{2b(x-bt-c_1)} & \text{for } b < 0; \\ 1 & \text{for } b \ge 0. \end{cases}$$

3.2. Square-root boundaries.

Theorem 6. *If* $g(t) = a\sqrt{t+\gamma} + c_0$, $f(t) = b\sqrt{t+\gamma} + c_0$, $\forall t \in (-\gamma; T]$, $b>a$, $T+\gamma > 0$, *then* $v_g(t,x) = v_1\left(t,\ \frac{x-c_0-a\sqrt{t+\gamma}}{(b-a)\sqrt{t+\gamma}}\right)$, $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma),\ x\right)$, $v_2(t,x) = v_3\left(-t+\frac{1}{2}\ln(T+\gamma),\ x\right)$, $v_3(t,x) = L^{-1}\left[V(p,x)\right]$, $V(p,x) = W(p, 1-x)$;

$$\alpha_0 = 0,\ \ \alpha_1 = 1,\ \ \alpha_{n+2} = \frac{-b(b-a)}{n+2}\,\alpha_{n+1} + \frac{(b-a)^2(n+p)}{(n+2)(n+1)}\,\alpha_n\ ,\ \forall n \in \mathbb{N}_0\,;$$

$$W(p,x) = c\,.\sum_{n=1}^{\infty} \alpha_n x^n,\ \ c = \frac{1}{p\,.\sum\limits_{n=1}^{\infty}\alpha_n}\ .$$

Corollary 6. *If* $g(t) = a\sqrt{t+\gamma} + c_0$ *and* $f(t) = c_0$, $\forall t \in (-\gamma; T]$, $a < 0$, $T+\gamma > 0$, *then* $v_g(t,x) = v_1\left(t,\ \frac{x-c_0-a\sqrt{t+\gamma}}{-a\sqrt{t+\gamma}}\right)$, $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma), x\right)$, $v_2(t,x) = v_3\left(-t+\frac{1}{2}\ln(T+\gamma),\ x\right)$, $v_3(t,x) = L^{-1}\left[V(p,x)\right]$, $V(p,x) = W(p, 1-x)$;

$$\beta_0 = 1,\ \beta_m = \frac{(-a)^{2m}}{(2m+1)!}\prod_{\widetilde{m}=1}^{m}(2\widetilde{m}-1+p), \forall m \in \mathbb{N};\ W(p,x) = c\,.\sum_{m=0}^{\infty}\beta_m x^{2m+1},$$

$$c = \frac{1}{p\,.\sum\limits_{m=0}^{\infty}\beta_m}\ .$$

4. Numerical experiments

The formulae in this paper were programmed and tabulated. The results were compared with the values of the crossing probabilities calculated by means of the Monte Carlo method and dynamical programming. The idea of the last method is to calculate the crossing probabilities, beginning from the horizon and moving to the starting moment step by step; for each t an array of probabilities is calculated using the array of the previous step.

The three results agree with one another. So we have a numerical support of our formulae besides the theoretical one. The algorithm that uses the formulae is the fastest one.

The functions $V(p,x)$ and $W(p,x)$ were written in the form of infinite power series. Alternatively, they can be expressed in terms of some special functions, for instance, the parabolic cylinder function (cf. [10]). For computational reasons, we preferred power series to special functions as well as a sequence of simple substitutions to a single compound substitution.

5. Conclusion

The propositions above give a comprehensive answer to the question about the crossing probabilities in two special cases: rectilinear parallel boundaries and square-root boundaries with a time-horizon. The obtained formulae are suitable for programming: calculations based on them are fast enough.

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ВЪРХУ ВЕРОЯТНОСТТА ЗА ДОСТИГАНЕ НА ПОДВИЖНА ГРАНИЦА ОТ БРАУНОВО ДВИЖЕНИЕ

Добромир Кралчев

Резюме. Разглеждаме вероятността Брауново движение да достигне подвижна двустранна граница до определен момент. В някои частни случаи извеждаме формули за тази вероятност.