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AN ABJ-CONNECTION ON ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

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Abstract. In a generalized B-manifold M, with an almost complex structure J and a Norden metric g we introduce an affine connection by the condition

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + g_{ij}a^k + J_{ij}b^k + \frac{1}{2}J_i^k\tilde{a}_j - \frac{1}{2}\delta_i^k\tilde{b}_j + \frac{1}{2}J_j^k\tilde{a}_i - \frac{1}{2}\delta_j^k\tilde{b}_i.$$

In this equation $\overline{\Gamma}$ and Γ are the Christoffel symbols of $\overline{\nabla}$ and of the connection ∇ of g respectively. We get some properties of the transformation defined by the above equation.

Key words: almost complex manifolds, Norden metric
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1. Introduction

We consider a manifold $M(dimM = 2n \geq 4)$ in the class GB of the generalized B-manifolds with a metric g and an additional structure J such that:

$$J^2 = -id, \quad g(Jx, Jy) = -g(x, y), \quad x, y \in \chi M. \tag{1.1}$$

So J is an almost complex structure and g is a Norden metric with respect to J [3]. Let ∇ be the Riemannian connection of g and R be the curvature tensor field of ∇.

The manifold M is in the class $SKN \subset GB$[1], if the structure J and the metric g satisfy (1.1) and

$$\nabla_i J_k^i = 0, \tag{1.2}$$

where J_k^i are the local coordinates of J.

The manifold M is in the class $AB \subset GB$ of the almost B-manifolds [5], if the structure J and the metric g satisfy (1.1) and

$$\nabla_i J_{jk} + \nabla_j J_{ki} + \nabla_k J_{ij} = 0, \tag{1.3}$$

where $J_{ij} = J_i^a g_{aj}$.

The manifold M is in the class $B \subset GB$ [7], if the structure J and the metric g satisfy (1.1) and

$$\nabla J = 0. \tag{1.4}$$

It is known that $B \subset AB \subset SKN \subset GB$ [5], [6].

Definition 1.1. The linear connection $\overline{\nabla}$ on a manifold with an almost complex structure J is a J-connection if $\overline{\nabla} J = 0$.

In [4] such a connection is called a B-connection.

In this paper we generalized the idea for a J-connection on a B-manifold, defined in [2] and we give

Definition 1.2. The linear connection $\overline{\nabla}$ on $M \in GB$ is an ABJ-connection if $\overline{\nabla}$ satisfy

$$\overline{\nabla}_i J_{jk} + \overline{\nabla}_j J_{ki} + \overline{\nabla}_k J_{ij} = 0.$$

Obviously, if $M \in AB$, then the Riemannian connection ∇ is an ABJ-connection. If $M \in B$, then ∇ is a J-connection.

2. An ABJ-connection

Theorem 2.1. *Let M be an almost B-manifold, ∇ be the Riemannian connection of g and Γ_{ij}^k be the Christoffel symbols of ∇. If a and b are arbitrary smooth vector fields on M, then the connection $\overline{\nabla}$, defined by the relation*

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + g_{ij} a^k + J_{ij} b^k + \frac{1}{2} J_i^k \tilde{a}_j - \frac{1}{2} \delta_i^k \tilde{b}_j + \frac{1}{2} J_j^k \tilde{a}_i - \frac{1}{2} \delta_j^k \tilde{b}_i, \tag{2.1}$$

where $\tilde{a}_i = J_i^t a_t$, $\tilde{b}_i = J_i^t b_t$, is a symmetric ABJ-connection.

Proof. We see that $\overline{\Gamma}_{ij}^k = \overline{\Gamma}_{ji}^k$, thus $\overline{\nabla}$ is a symmetric connection. Now we find the covariant derivative $\overline{\nabla}$ of J_{ij} by using the well known formula

$$\overline{\nabla}_i J_{jk} = \partial_i J_{jk} - \overline{\Gamma}_{ij}^a J_{ak} - \overline{\Gamma}_{ik}^a J_{aj}$$

and from (2.1), we get

$$\overline{\nabla}_i J_{jk} = \nabla_i J_{jk} - \frac{1}{2} g_{ij}\tilde{a}_k - \frac{1}{2} g_{ik}\tilde{a}_j + g_{jk}\tilde{a}_i - \frac{1}{2} J_{ij}\tilde{b}_k - \frac{1}{2} J_{ik}\tilde{b}_j + J_{jk}\tilde{b}_i.$$

After direct calculations we obtain $\overline{\nabla}_i J_{jk} + \overline{\nabla}_j J_{ki} + \overline{\nabla}_k J_{ij} = 0$. So $\overline{\nabla}$ is an ABJ-connection. □

Corollary 2.2. *Let M be in the class AB also ∇ and $\overline{\nabla}$ satisfy (2.1). If $\overline{\nabla}$ is a J-connection, then ∇ is a J-connection too.*

Proof. From

$$\overline{\nabla}_i J_j^k = \partial_i J_j^k - \overline{\Gamma}_{ij}^a J_a^k + \overline{\Gamma}_{ia}^k J_j^a$$

and using (2.1), we get

$$\overline{\nabla}_i J_j^k = \nabla_i J_j^k - g_{ij}(\tilde{a}^k + b^k) + J_{ij}(a^k - \tilde{b}_k) + \frac{1}{2}\delta_i^k(b_j + \tilde{a}_j) - \frac{1}{2} J_i^k(a_j - \tilde{b}_j). \tag{2.2}$$

Let us assume, that $\overline{\nabla}_i J_j^k = 0$. Then (2.2) implies

$$\nabla_i J_j^k = g_{ij}(\tilde{a}^k + b^k) - J_{ij}(a^k - \tilde{b}_k) - \frac{1}{2}\delta_i^k(b_j + \tilde{a}_j) + \frac{1}{2} J_i^k(a_j - \tilde{b}_j). \tag{2.3}$$

In (2.3) we contract with $k = i$ and we get

$$\nabla_i J_j^i = -n(\tilde{a}_j + b_j).$$

The equation (1.3) implies (1.2) and then $\tilde{a}_j = -b_j$. After substituting the last result in (2.3), we obtain $\nabla_i J_j^k = 0$, i.e. ∇ is a J-connection. □

3. The case $a = 0$

Let M be in AB also $\overline{\nabla}$ and ∇ satisfy (2.1). If $a_k = \tilde{a}_k = 0$, then (2.1) has the form

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + T_{ij}^k, \quad T_{ij}^k = J_{ij} b^k - \frac{1}{2}\delta_i^k \tilde{b}_j - \frac{1}{2}\delta_j^k \tilde{b}_i. \tag{3.1}$$

For the curvature tensor fields $\overline{R}$ of $\overline{\nabla}$ and R of ∇ it is well known the identity

$$\overline{R}_{ijk}^h = R_{ijk}^h + \nabla_j T_{ik}^h - \nabla_k T_{ij}^h + T_{ik}^s T_{sj}^h - T_{ij}^s T_{sk}^h. \tag{3.2}$$

From (3.1) and (3.2) we obtain

$$
\begin{aligned}
\overline{R}^h_{ijk} &= R^h_{ijk} + J_{ik}(\nabla_j b^h + \tilde{b}_j b^h - \frac{1}{2}\delta^h_j \tilde{b}^s b_s) \\
&- J_{ij}(\nabla_k b^h + \tilde{b}_k b^h - \frac{1}{2}\delta^h_k \tilde{b}^s b_s) \\
&- \frac{1}{2}\delta^h_i(\nabla_j \tilde{b}_k - \nabla_k \tilde{b}_j) - \frac{1}{4}\delta^h_k(2\nabla_j \tilde{b}_i + \tilde{b}_i\tilde{b}_j) \\
&+ \frac{1}{4}\delta^h_j(2\nabla_k \tilde{b}_i + \tilde{b}_i\tilde{b}_k) - b^h(\nabla_k J_{ij} - \nabla_j J_{ik}).
\end{aligned}
\tag{3.3}
$$

Theorem 3.1. *Let M be in AB also $\overline{\nabla}$ and ∇ satisfy (3.1). Then $\overline{\nabla}$ is an equiaffine connection if and only if the vector field $\tilde{b}$ is gradient.*

Proof. We know ∇ is an equiaffine connection, i.e. $R^i_{ijk} = 0$. That's why contracting (3.3) with $h = i$, we get

$$\overline{R}^i_{ijk} = (n - \frac{1}{2})(\nabla_k \tilde{b}_j - \nabla_j \tilde{b}_k).$$

Then $\overline{R}^i_{ijk} = 0$ if and only if $\nabla_k \tilde{b}_j = \nabla_j \tilde{b}_k$. The last condition implies the vector field $\tilde{b}$ is gradient. □

Theorem 3.2. *Let M be a B-manifold and ∇ be the Riemannian connection of g. If b is a smooth vector field on M, and $\overline{\nabla}$ is a locally flat ABJ-connection, defined by the relation (3.1), then b is an isotropic vector field and $\overline{\nabla}$ is a locally flat connection.*

Proof. We consider $M \in B$, i.e. ∇ is a J-connection.
We denote

$$P_{kh} = \nabla_k b_h + \tilde{b}_k b_h - \frac{\varphi}{2} g_{kh}, \quad \varphi = \tilde{b}^s b_s, \quad \tilde{P}_{kh} = J^t_h P_{kt}. \tag{3.4}$$

From (1.4) it is follows

$$\tilde{P}_{kh} = \nabla_k \tilde{b}_h + \tilde{b}_k \tilde{b}_h - \frac{\varphi}{2} J_{kh}. \tag{3.5}$$

From $R^i_{ijk} = 0$ we get $\nabla_k \tilde{b}_j = \nabla_j \tilde{b}_k$, so $\tilde{b}$ is a gradient vector. We assume, that $\overline{\nabla}$ is a locally flat connection and it is necessary and sufficient that $\overline{R} = 0$. By using the last condition, (1.4), (3.3), (3.4), (3.5) and lowering the index h in (3.3), we have

$$R_{hijk} = J_{ij} P_{kh} - J_{ik} P_{jh} + \frac{1}{4} g_{kh} Q_{ji} - \frac{1}{4} g_{jh} Q_{ki}, \tag{3.6}$$

where

$$Q_{ij} = 2\tilde{P}_{ji} - \tilde{b}_i\tilde{b}_j + \varphi J_{ij}. \tag{3.7}$$

With the help of the identity $R_{hijk} = R_{jkhi}$ and the equation (3.6) we find

$$J_{ik}P_{jh} - J_{ij}P_{kh} - J_{ki}P_{hj} + J_{kh}P_{ij} = \frac{1}{4}g_{kh}Q_{ij} - \frac{1}{4}g_{ij}Q_{kh}. \tag{3.8}$$

We exchange k and h in (3.8) and by using the relation $Q_{ij} = Q_{ji}$, we get

$$J_{ik}(P_{hj} - P_{jh}) + J_{ij}(P_{kh} - P_{hk}) + J_{ih}(P_{jk} - P_{kj}) = 0.$$

In the last equation we contract with J^{ij} and we obtain:

$$P_{kh} = P_{hk}. \tag{3.9}$$

We substitute (3.9) in (3.8) and we contract with g^{kh}, then we get

$$Q_{ij} = \frac{Q}{2n}g_{ij} - \frac{2P}{n}J_{ij}; \qquad Q = Q_s^s, \quad P = P_s^s. \tag{3.10}$$

If we denote $\tilde{Q}_{ij} = J_j^sQ_{is}$, then from (3.10) we see, that $\tilde{Q}_{ij} = \tilde{Q}_{ji}$.

On the other hand from (3.7) we have

$$\tilde{Q}_{ij} = -2P_{ji} + \tilde{b}_ib_j - \varphi g_{ij}.$$

The last equations and (3.9) imply

$$\tilde{b}_ib_j = \tilde{b}_jb_i. \tag{3.11}$$

We contract with b^i and then with $\tilde{b}^j$ in (3.11) and we get $(\varphi)^2 + (\phi)^2 = 0$, $\phi = b_ib^i$. Then $\varphi = \phi = 0$. So we prove that b is an isotropic vector.

Transvecting (3.8) with J^{kh}, we obtain

$$P_{ij} = \frac{1}{2n}(\frac{\tilde{Q}}{4}g_{ij} - \tilde{P}J_{ij}); \qquad \tilde{Q} = \tilde{Q}_s^s, \quad \tilde{P} = \tilde{P}_s^s. \tag{3.12}$$

We contract (3.12) with g^{ij} and (3.7) with J^{ij} and we find the system

$$\tilde{Q} = 4P, \qquad \tilde{Q} = -2P, \tag{3.13}$$

and it's decision is $\tilde{Q} = P = 0$.

From (3.10), (3.12), the last condition and (3.6) we get

$$R_{hijk} = \frac{\tilde{P}}{2n}(J_{ik}J_{jh} - J_{ij}J_{kh} + \frac{1}{2}g_{kh}g_{ij} - \frac{1}{2}g_{ik}g_{jh}). \tag{3.14}$$

Let S be the Ricci tensor of ∇. Transvecting with g^{ij} in (3.14), we obtain:

$$S_{hk} = (n - \frac{3}{2})\frac{\tilde{P}}{2n}g_{kh}. \tag{3.15}$$

For the scalar curvature $\tau = S_{ij}g^{ij}$ we have

$$\tau = (n - \frac{3}{2})\tilde{P}. \tag{3.16}$$

If (M, g, J) is in the class B, then $R_{hijk}J^{ij} = S_{aj}J^a_k$. Thus, it follows $R_{hijk}J^{ij}J^{hk} = -\tau$. Then from (3.14) we find

$$\tau = (2n - \frac{3}{2})\tilde{P}. \tag{3.17}$$

Collecting the system (3.16), (3.17), we get $\tilde{P} = \tau = 0$. After substituting the last results in (3.14), we obtain $R = 0$. So the theorem is proved. □

Corollary 3.3. *Let M and $\overline{M}$ satisfy the conditions of Theorem 3.2 and $\alpha = \{b, \tilde{b}\}$ be the two-dimensional section in T_pM, $p \in M$. Then for two arbitrary vectors $x, y \in \alpha$ we have $g(x,x) = g(x,y) = g(y,y) = 0$.*

4. The case $b = 0$

Let M be in AB also $\overline{\nabla}$ and ∇ satisfy (2.1). If $b_k = \tilde{b}_k = 0$, then (2.1) has the form

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + T^k_{ij}, \qquad T^k_{ij} = g_{ij}a^k + \frac{1}{2}J^k_i\tilde{a}_j + \frac{1}{2}J^k_j\tilde{a}_i, \tag{4.1}$$

For the curvature tensor fields $\overline{R}$ of $\overline{\nabla}$ and R of ∇ we get

$$\begin{aligned} \overline{R}^h_{ijk} = R^h_{ijk} &+ g_{ik}P^h_j - g_{ij}P^h_k + \frac{1}{4}J^h_kQ_{ji} - \frac{1}{4}J^h_jQ_{ki} \\ &- \frac{1}{2}J_{ik}\tilde{a}_ja^h + \frac{1}{2}J_{ij}\tilde{a}_ka^h - \frac{1}{4}\delta^h_k\tilde{a}_i\tilde{a}_j + \frac{1}{4}\delta^h_j\tilde{a}_i\tilde{a}_k \\ &- \frac{1}{2}J^h_i(\nabla_k\tilde{a}_j - \nabla_j\tilde{a}_k), \end{aligned} \tag{4.2}$$

$$P_{kh} = \nabla_k a_h + a_k a_h + \frac{1}{2}\tilde{a}_h \tilde{a}_k + \frac{\varphi}{2} J_{kh}, \quad \varphi = \tilde{a}^s a_s. \tag{4.3}$$

$$Q_{kh} = 2\nabla_k \tilde{a}_h + \tilde{a}_k a_h + \tilde{a}_h a_k. \tag{4.4}$$

Theorem 4.1. *Let M be in AB also $\overline{\nabla}$ and ∇ satisfy (4.1). Then $\overline{\nabla}$ is an equiaffine connection if and only if the vector field a is gradient.*

Proof. We know ∇ is an equiaffine connection, i.e. $R^i_{ijk} = 0$. That's why contracting (3.3) with $h = i$, we get

$$\overline{R}^i_{ijk} = \frac{1}{2}(\nabla_j a_k - \nabla_k a_j).$$

Then $\overline{R}^i_{ijk} = 0$ if and only if $\nabla_k a_j = \nabla_j a_k$. The last condition implies the vector field a is gradient. □

Theorem 4.2. *Let M be a B-manifold, ∇ be the Riemannian connection of g, R be the curvature tensor field of ∇. If a is a smooth vector field on M, and $\overline{\nabla}$ is a locally flat ABJ-connection, defined by (4.1), then a is gradient vector field and ∇ is a locally flat connection.*

Proof. We consider $M \in B$, i.e. ∇ is a J-connection. We assume, that $\overline{\nabla}$ is a locally flat connection and it is necessary and sufficient that $\overline{R} = 0$.

If we denote $\tilde{P}_{kh} = J^t_h P_{kt}$, $\tilde{Q}_{kh} = J^t_h Q_{kt}$, from (1.4) it is follows:

$$\tilde{P}_{kh} = \nabla_k \tilde{a}_h + a_k \tilde{a}_h - \frac{1}{2} a_h \tilde{a}_k - \frac{\varphi}{2} g_{kh}, \quad \tilde{Q}_{kh} = -2\nabla_k a_h + \tilde{a}_k \tilde{a}_h - a_h a_k. \tag{4.5}$$

From $R^i_{ijk} = 0$ we get $\nabla_k a_j = \nabla_j a_k$, so a is a gradient vector. By using the last condition and from (4.3),(4.4), (4.5) we have $P_{kh} = P_{hk}$, $\tilde{Q}_{kh} = \tilde{Q}_{hk}$. With the help of the identity $R_{hijk} = R_{jkhi}$ and the equation (4.2),(4.3),(4.4) using the same ideas like in the previous paragraph we get

$$Q_{kh} - Q_{hk} = 2(\tilde{a}_h a_k - \tilde{a}_k a_h). \tag{4.6}$$

From (4.2) and using (4.6) we find $Q_{kh} = Q_{hk}$. Then $\tilde{a}_k a_h = a_h \tilde{a}_k$. The last condition implies, that $a^2 = 0$, so a is an isotropic vector.

If M is in the class B, then for the scalar curvature $\tau = S_{ij} g^{ij}$ it is known that $R_{hijk} J^{ij} J^{hk} = -\tau$. From the last condition after long calculations, we obtain $\tilde{Q}^s_s = \tilde{P}^s_s = Q^s_s = P^s_s = 0$ and also $\tau = \tau^* = 0$. Then $R = 0$. □

Corollary 4.3. *Let M and $\overline{M}$ satisfy the conditions of Theorem 4.2 and $\alpha = \{b, \tilde{b}\}$ be the two-dimensional section in T_pM, $p \in M$. Then for two arbitrary vectors $x, y \in \alpha$ we have $g(x,x) = g(x,y) = g(y,y) = 0$.*

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ЕДНА ABJ-СВЪРЗАНОСТ ВЪРХУ ПОЧТИ КОМПЛЕКСНО МНОГООБРАЗИЕ С НОРДЕНОВА МЕТРИКА

Ива Докузова

Резюме. В обобщено B-многообразие M с почти комплексна структура J и норденова метрика g дефинираме една линейна свързаност с условието

$$\overline{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + g_{ij}a^k + J_{ij}b^k + \frac{1}{2}J_i^k\tilde{a}_j - \frac{1}{2}\delta_i^k\tilde{b}_j + \frac{1}{2}J_j^k\tilde{a}_i - \frac{1}{2}\delta_j^k\tilde{b}_i.$$

Тук $\overline{\Gamma}$ и Γ са съответно символите на Кристофел за $\overline{\nabla}$ и за римановата свързаност ∇ на g. В настоящата работа ние намираме някои свойства на изображението, зададено с горното уравнение.