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ON SOME SPECIAL COMPOSITIONS AND CURVATURE PROPERTIES ON A THREE-DIMENSIONAL WEYL SPACE

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Abstract. Special compositions, generated by a net in a space with a symmetric linear connection are considered in [2], [3] and [5]. In this paper, the special compositions generated by a net in the 3-dimensional Weyl space are characterized in terms of the prolonged covariant differentiation. Some equations and applications of the curvature tensor and the Ricci tensor on a 3-dimensional Weyl space are given.

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***Key words*:** net, Weyl space, composition, prolonged differentiation, Weyl connection.

1. Preliminaries

Let W_3 be a 3-dimensional Weyl space with metric tensor g_{ik} and its inverse tensor g^{kj}, i.e. $g_{ik}g^{kj} = \delta_i^j,\ i,j,k = 1,2,3.$

There is known [6], the Weyl connection ∇ with components Γ_{ij}^k is determined by the equation

$$(1) \qquad \Gamma_{ij}^k = \left\{ {k \atop ij} \right\} - \Big(\omega_i\delta_j^k + \omega_j\delta_i^k - g_{ij}g^{ks}\omega_s\Big),$$

where ω_k is the complementary vector of W_3 and $\left\{ {k \atop ij} \right\}$ are the Cristoffel symbols, determined by g_{ij}. There are valid the equations

$$(2) \qquad \nabla_k g_{ij} = 2\omega_k g_{ij},\ \nabla_k g^{ij} = -2\omega_k g^{ij}.$$

Let us consider a composition $W_3(X_2 \times X_1)$ in W_3, where $X_2(\dim X_2 = 2)$, $X_1(\dim X_1 = 1)$ are the fundamental manifolds. There exists a unique position of each of the fundamental manifolds X_2 and X_1 at every point $p \in W_3$, which is denoted by $P(X_2)$ and $P(X_1)$, respectively.

According to [9], W_3 is the space of the composition $W_3(X_2 \times X_1)$, if there exists a tensor field a_i^j of type (1,1) determined by the equations

$$a_i^j a_j^k = \delta_i^k, \tag{3}$$

$$N_{ij}^k = a_i^s \nabla_s a_j^k - a_j^s \nabla_s a_i^k - a_s^k (\nabla_i a_j^s - \nabla_j a_i^s) = 0, \tag{4}$$

where N_{ij}^k is the Nijenhuis tensor of the structure a_i^j.

The projecting tensors $\overset{n}{a}{}_i^k$ and $\overset{m}{a}{}_i^k$ have the form

$$\overset{n}{a}{}_i^k = \frac{1}{2}(\delta_i^k + a_i^k), \ \overset{m}{a}{}_i^k = \frac{1}{2}(\delta_i^k - a_i^k), \tag{5}$$

where, because of (3), it follows the properties

$$\overset{n}{a}{}_i^k \overset{n}{a}{}_k^s = \overset{n}{a}{}_i^s, \ \overset{m}{a}{}_i^k \overset{m}{a}{}_k^s = \overset{m}{a}{}_i^s, \ \overset{n}{a}{}_i^k \overset{m}{a}{}_k^s = \overset{m}{a}{}_i^k \overset{n}{a}{}_k^s = 0.$$

Following [7] and [9], the composition $W_3(X_2 \times X_1)$ is called geodesic-Chebyshevian, if the tangent section of $P(X_2)$ and the tangent vector of $P(X_1)$ are translated parallelly in the direction of every curve of $P(X_2)$. The characteristic of the geodesic-Chebyshevian composition is

$$\overset{n}{a}{}_i^k \nabla_k \overset{n}{a}{}_j^s = 0. \tag{6}$$

A composition $W_3(X_2 \times X_1)$ is called Chebyshevian-geodesic, if the tangent section of $P(X_2)$ is translated parallelly in the curve $P(X_1)$ and the curve $P(X_1)$ is geodesic. The characteristic of the Chebyshevian-geodesic composition is

$$\overset{m}{a}{}_i^k \nabla_k \overset{m}{a}{}_j^s = 0. \tag{7}$$

Let $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ be a net in W_3, determined by independent tangent vector fields $\underset{k}{v}^i$ of the curve of the net $(k = 1, 2, 3)$. We determine the inverse covectors

$\overset{k}{v}_i$ of $\underset{k}{v}^i$ $(k = 1, 2, 3)$, respectively, by the equations

$$\underset{i}{v}^k \overset{s}{v}_k = \delta_i^s \Leftrightarrow \underset{i}{v}^k \overset{i}{v}_s = \delta_s^k. \tag{8}$$

According to [5], the prolonged covariant differentiation $\overset{\circ}{\nabla}$ of the satellite A with weight $\{p\}$ in the Weyl space is defined by

$$\overset{\circ}{\nabla}_i A = \nabla_i A - p\omega_i A. \tag{9}$$

Having in mind that the weights on the affinor a_i^j, the vector $\underset{s}{v}^j$ and the covector $\overset{s}{v}_j$ are $\{0\}, \{-1\}$ and $\{+1\}$, respectively, and using (9), we obtain

$$\overset{\circ}{\nabla}_k a_i^j = \nabla_k a_i^j; \tag{10}$$

$$\overset{\circ}{\nabla}_k \underset{s}{v}^j = \nabla_k \underset{s}{v}^j + \omega_k \underset{s}{v}^j; \tag{11}$$

$$\overset{\circ}{\nabla}_k \overset{s}{v}_j = \nabla_k \overset{s}{v}_j - \omega_k \overset{s}{v}_j. \tag{12}$$

In [5] there are found the derivative equations of the directional vectors of the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$:

$$\overset{\circ}{\nabla}_i \underset{k}{v}^s = \underset{k}{\overset{r}{T}}_i \underset{r}{v}^s, \ \overset{\circ}{\nabla}_i \overset{k}{v}_s = -\underset{r}{\overset{k}{T}}_i \overset{r}{v}_s, \ k = 1, 2, 3. \tag{13}$$

In [2] is given the form of the curvature tensor on a 3-dimensional Weyl space, i.e.

$$R_{ijk}^s = \frac{1}{3}\Big\{\Big(g_{jk}S_{il} - g_{ik}S_{jl}\Big)g^{ls} + S_{jk}\delta_i^s - S_{ik}\delta_j^s + \Big(S_{ji} - S_{ij}\Big)\delta_k^s\Big\}, \tag{14}$$

where $S_{jk} = 2R_{jk} + R_{kj} - \frac{3R}{4}g_{jk}$ and $R = g^{ij}R_{ij}$ is the scalar curvature. According to (1) and the identity for the curvature tensor of a Weyl space [6], we have the following equality for $n = 3$:

$$\nabla_j \omega_i - \nabla_i \omega_j = \frac{R_{ji} - R_{ij}}{3}. \tag{15}$$

2. Some special compositions in W_3

In this section we give geometric characteristics for the geodesic-Chebyshevian and the Chebyshevian-geodesic compositions.

In [2] there is defined the affinor a_i^k of the composition in the Weyl space. It is determined uniquely by the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ and it has the following form in W_3:

$$a_i^k = \underset{1}{v}^k \overset{1}{v}_i + \underset{2}{v}^k \overset{2}{v}_i - \underset{3}{v}^k \overset{3}{v}_i. \tag{16}$$

There follows immediately that a_i^k satisfies (3) and the conditions

$$a_k^s \underset{1}{v}^k = \underset{1}{v}^s,\ a_k^s \underset{2}{v}^k = \underset{2}{v}^s,\ a_k^s \underset{3}{v}^k = -\underset{3}{v}^s. \tag{17}$$

According to (5) and (16), for the projecting tensors we have

$$\overset{n}{a}{}_i^k = \underset{1}{v}^k \overset{1}{v}_i + \underset{2}{v}^k \overset{2}{v}_i,\ \overset{m}{a}{}_i^k = \underset{3}{v}^k \overset{3}{v}_i. \tag{18}$$

The composition $W_3(X_2 \times X_1)$ is determined by a_i^k, if the affinor satisfies (4). The composition $W_3(X_2 \times X_1)$ is called associated to the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$.

Theorem 1. *The composition $W_3(X_2 \times X_1)$ associated to the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ is geodesic-Chebyshevian if and only if the coefficients of the derivative equations $\underset{3}{\overset{1}{T}}_k, \underset{3}{\overset{2}{T}}_k, \underset{1}{\overset{3}{T}}_k, \underset{2}{\overset{3}{T}}_k$ belong to $P(X_1)$, i.e.*

$$\underset{3}{\overset{1}{T}}_k = a\overset{k}{v}_3,\ \underset{3}{\overset{2}{T}}_k = b\overset{k}{v}_3,\ \underset{1}{\overset{3}{T}}_k = c\overset{k}{v}_3,\ \underset{2}{\overset{3}{T}}_k = d\overset{k}{v}_3,\ a, b, c, d, \in \mathbb{R} \tag{19}$$

Proof. According to (10), the condition (6) has the form $\overset{n}{a}{}_j^k \overset{\circ}{\nabla}_k \overset{n}{a}{}_i^s = 0$. Having in mind (13), (18) and the linear independence of the vectors $\underset{1}{v}^k$, $\underset{2}{v}^k$, $\underset{3}{v}^k$, we obtain the system

$$\begin{aligned}
&\overset{n}{a}{}_j^k\left(-\underset{l}{\overset{1}{T}}_k \overset{l}{v}_i + \underset{1}{\overset{1}{T}}_k \overset{1}{v}_i + \underset{2}{\overset{1}{T}}_k \overset{2}{v}_i\right) = 0,\\
&\overset{n}{a}{}_j^k\left(-\underset{l}{\overset{2}{T}}_k \overset{l}{v}_i + \underset{1}{\overset{2}{T}}_k \overset{1}{v}_i + \underset{2}{\overset{2}{T}}_k \overset{2}{v}_i\right) = 0,\\
&\overset{n}{a}{}_j^k\left(\underset{1}{\overset{3}{T}}_k \overset{1}{v}_i + \underset{2}{\overset{3}{T}}_k \overset{2}{v}_i\right) = 0,
\end{aligned} \tag{20}$$

Using (18), we receive the following equality by contracting the last equation of (20) with $\underset{1}{v}^i$ and $\underset{2}{v}^i$:

$$\overset{1}{v}_j \overset{3}{\underset{1}{T}}_k \underset{1}{v}^k + \overset{2}{v}_j \overset{3}{\underset{1}{T}}_k \underset{2}{v}^k = 0, \ \overset{1}{v}_j \overset{3}{\underset{2}{T}}_k \underset{1}{v}^k + \overset{2}{v}_j \overset{3}{\underset{2}{T}}_k \underset{2}{v}^k = 0.$$

Then, because of the linear independence of the covectors $\overset{1}{v}_j$ and $\overset{2}{v}_j$, we have

$$\overset{3}{\underset{1}{T}}_k \underset{1}{v}^k = \overset{3}{\underset{1}{T}}_k \underset{2}{v}^k = \overset{3}{\underset{2}{T}}_k \underset{1}{v}^k = \overset{3}{\underset{2}{T}}_k \underset{2}{v}^k = 0. \tag{21}$$

The equations (21) mean that the covectors $\overset{3}{\underset{1}{T}}_k$ and $\overset{3}{\underset{2}{T}}_k$ belong to the position $P(X_1)$, i.e. they are collinear to the covector $\overset{3}{v}_k$. Having in mind the first and the second equations of (20), using the linear independence of the covectors $\overset{1}{v}_j$ and $\overset{2}{v}_j$, we find

$$\overset{2}{\underset{3}{T}}_k \underset{1}{v}^k = \overset{2}{\underset{3}{T}}_k \underset{2}{v}^k = \overset{1}{\underset{3}{T}}_k \underset{1}{v}^k = \overset{1}{\underset{3}{T}}_k \underset{2}{v}^k = 0. \tag{22}$$

The equations (22) imply that the covectors $\overset{1}{\underset{3}{T}}_k$ and $\overset{2}{\underset{3}{T}}_k$ are collinear to the covector $\overset{3}{v}_k$, i.e. they belong to the position $P(X_1)$. Hence, (21) and (22) imply (19). □

Let the composition $W_3(X_2 \times X_1)$ be geodesic-Chebyshevian and the curves $\underset{1}{v}$ and $\underset{2}{v}$ are geodesic. According to [4], we have the conditions

$$\underset{1}{v}^k \nabla_k \underset{1}{v}^s = \underset{2}{v}^k \nabla_k \underset{2}{v}^s = 0, \tag{23}$$

where ∇ is the Weyl connection. In this case, we verify immediately that an arbitrary vector of the section $(\underset{1}{v}^k, \underset{2}{v}^k)$ is translated parallelly of an arbitrary curve of $P(X_2)$. Since $\underset{3}{v}^s$ is translated parallelly in the direction of every curve of $P(X_2)$ then $\underset{3}{v}^s$ is translated parallelly of the curves $\underset{1}{v}$ and $\underset{2}{v}$. Then we have:

$$\underset{1}{v}^k \nabla_k \underset{3}{v}^s = \underset{2}{v}^k \nabla_k \underset{3}{v}^s = 0. \tag{24}$$

Having in mind (11), (13) and (19), we find

$$\nabla_k \underset{3}{v}^j = \overset{3}{v}_k \Big(a\underset{1}{v}^j + b\underset{2}{v}^j \Big) + \Big(\underset{3}{\overset{3}{T}}_k - \omega_k \Big) \underset{3}{v}^j. \tag{25}$$

After the contracting of (25) consecutively by the vectors $\underset{1}{v}^k$ and $\underset{2}{v}^k$, because of the condition (24), we receive

$$\underset{3}{\overset{3}{T}}_k - \omega_k = e\overset{3}{v}_k, \ \ e \in \mathbb{R}. \tag{26}$$

By analogy, using (12), (13), (19) and (23), we find

$$\begin{aligned}
&\underset{1}{\overset{1}{T}}_k - \omega_k = a_1\overset{2}{v}_k + b_1\overset{3}{v}_k, \ \underset{2}{\overset{2}{T}}_k - \omega_k = a_4\overset{1}{v}_k + b_4\overset{3}{v}_k,\\
&\underset{1}{\overset{2}{T}}_k = a_2\overset{2}{v}_k + b_2\overset{3}{v}_k,\\
&\underset{2}{\overset{1}{T}}_k = a_3\overset{1}{v}_k + b_3\overset{3}{v}_k.
\end{aligned} \tag{27}$$

Hence the following proposition is valid:

Theorem 2. *Let the composition $W_3(X_2 \times X_1)$ be geodesic-Chebyshevian and the curves $\underset{1}{v}$ and $\underset{2}{v}$ on the position $P(X_2)$ be geodesics. Then all coefficients of the derivative equations (13) are determined by the equalities (19), (26) and (27).*

Corollary 1. *Let the composition $W_3(X_2 \times X_1)$ be geodesic-Chebyshevian and the curves $\underset{1}{v}$ and $\underset{2}{v}$ on $P(X_2)$ be geodesics. Then the derivative equations with respect to the Weyl connection have the form:*

$$\begin{aligned}
&\nabla_k \underset{3}{v}^j = \overset{3}{v}_k \Big(a\underset{1}{v}^j + b\underset{2}{v}^j + e\underset{3}{v}^j \Big), \ \nabla_k \overset{3}{v}_j = -\overset{3}{v}_k \Big(c\overset{1}{v}_j + d\overset{2}{v}_j + e\overset{3}{v}_j \Big),\\
&\nabla_k \underset{1}{v}^j = \overset{3}{v}_k \Big(b_1\underset{1}{v}^j + b_2\underset{2}{v}^j + c\underset{3}{v}^j \Big) + \overset{2}{v}_k \Big(a_1\underset{1}{v}^j + a_2\underset{2}{v}^j \Big),\\
&\nabla_k \overset{1}{v}_j = -\overset{3}{v}_k \Big(b_1\overset{1}{v}_j + b_3\overset{2}{v}_j + a\overset{3}{v}_j \Big) - a_3\overset{2}{v}_j\overset{1}{v}_k - a_1\overset{1}{v}_j\overset{2}{v}_k,\\
&\nabla_k \underset{2}{v}^j = \overset{3}{v}_k \Big(b_3\underset{1}{v}^j + b_4\underset{2}{v}^j + d\underset{3}{v}^j \Big) + \overset{1}{v}_k \Big(a_3\underset{1}{v}^j + a_4\underset{2}{v}^j \Big),\\
&\nabla_k \overset{2}{v}_j = -\overset{3}{v}_k \Big(b_2\overset{1}{v}_j + b_4\overset{2}{v}_j + b\overset{3}{v}_j \Big) - a_4\overset{2}{v}_j\overset{1}{v}_k - a_2\overset{1}{v}_j\overset{2}{v}_k.
\end{aligned} \tag{28}$$

Theorem 3. *The composition* $W_3(X_2 \times X_1)$ *associated to the net* $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ *is Chebyshevian-geodesic if and only if the coefficients of the derivative equations* $\overset{1}{\underset{3}{T}}_k, \overset{2}{\underset{3}{T}}_k, \overset{3}{\underset{1}{T}}_k, \overset{3}{\underset{2}{T}}_k$ *belong to* $P(X_2)$*, i.e. they are a linear combination of the covectors* $\overset{1}{v}_k$ *and* $\overset{2}{v}_k$.

Proof. According to (10), condition (7) for the Chebyshevian-geodesic composition has the form $\overset{m}{a}{}^k_j \overset{\circ}{\nabla}_k \overset{m}{a}{}^s_i = 0$. Having in mind (13), (18) and the linear independence of the vectors $\underset{1}{v}^k$, $\underset{2}{v}^k$, $\underset{3}{v}^k$, by analogy of the proof of Theorem 1, we obtain the system:

$$\overset{1}{\underset{3}{T}}_k \underset{3}{v}^k = \overset{2}{\underset{3}{T}}_k \underset{3}{v}^k = \overset{3}{\underset{1}{T}}_k \underset{3}{v}^k = \overset{3}{\underset{2}{T}}_k \underset{3}{v}^k = 0. \tag{29}$$

whence the coefficients $\overset{1}{\underset{3}{T}}_k$, $\overset{2}{\underset{3}{T}}_k$, $\overset{3}{\underset{1}{T}}_k$ and $\overset{3}{\underset{2}{T}}_k$ are a linear combination of the covectors $\overset{1}{v}_k$ and $\overset{2}{v}_k$. □

Theorem 4. *Let the composition* $W_3(X_2 \times X_1)$ *be Chebyshevian-geodesic. Then the coefficients of the derivative equations* $\overset{1}{\underset{1}{T}}_k$, $\overset{2}{\underset{2}{T}}_k$, $\overset{3}{\underset{3}{T}}_k$, $\overset{1}{\underset{2}{T}}_k$ *and* $\overset{2}{\underset{1}{T}}_k$ *are determined by the equations:*

$$\begin{aligned} &\overset{1}{\underset{1}{T}}_k - \omega_k = a_2\overset{1}{v}_k + b_2\overset{2}{v}_k, \ \overset{2}{\underset{2}{T}}_k - \omega_k = a_5\overset{1}{v}_k + b_5\overset{2}{v}_k, \\ &\overset{2}{\underset{1}{T}}_k = a_3\overset{1}{v}_k + b_3\overset{2}{v}_k, \ \overset{3}{\underset{3}{T}}_k - \omega_k = a_1\overset{1}{v}_k + b_1\overset{2}{v}_k, \\ &\overset{1}{\underset{2}{T}}_k = a_4\overset{1}{v}_k + b_4\overset{2}{v}_k, \end{aligned} \tag{30}$$

where ω_k *is the complementary vector of* W_3.

Proof. Since the curve $P(X_1)$ is geodesic, according to [4], we have:

$$\underset{3}{v}^k \nabla_k \underset{3}{v}^s = 0. \tag{31}$$

According to Theorem 3, for the coefficients $\overset{1}{\underset{3}{T}}_k, \overset{2}{\underset{3}{T}}_k, \overset{3}{\underset{1}{T}}_k$ and $\overset{3}{\underset{2}{T}}_k$ we have:

$$\overset{1}{\underset{3}{T}}_k = a\overset{1}{v}_k + b\overset{2}{v}_k, \ \overset{2}{\underset{3}{T}}_k = c\overset{1}{v}_k + d\overset{2}{v}_k, \ \overset{3}{\underset{1}{T}}_k = l\overset{1}{v}_k + h\overset{2}{v}_k, \ \overset{3}{\underset{2}{T}}_k = e\overset{1}{v}_k + f\overset{2}{v}_k. \tag{32}$$

Using (12), (13), (32) and condition (31) we find

$$\overset{3}{\underset{3}{T}}_k - \omega_k = a_1 \overset{1}{v}_k + b_1 \overset{2}{v}_k. \tag{33}$$

Since the position $P(X_2)$ is Chebyshevian then an arbitrary vector on $(\underset{1}{v}^k,\ \underset{2}{v}^k)$ is translated parallelly in the direction of a vector $\underset{3}{v}^k$. The following conditions are valid for the symmetric Weyl connection ∇ and the vectors $\underset{1}{v}^s$ and $\underset{2}{v}^s$ [4]:

$$\underset{3}{v}^k \nabla_k \underset{1}{v}^s = \underset{3}{v}^k \nabla_k \underset{2}{v}^s = 0. \tag{34}$$

Then using (11), (13), (32) and condition (34), we obtain the rest of the qualities in (30). □

3. The curvature properties of W_3

There is known [2], the curvature tensor is expressed by the Ricci tensor and the metric tensor for every 3-dimensional Weyl space, i.e. equation (14) is valid.

In this section we give complementary conditions for the curvature tensor and the Ricci tensor on W_3.

Theorem 5. *Let W_3 be a 3-dimensional Weyl space and ∇ be the Weyl connection on W_3. Then the Ricci tensor R_{jk}, the complementary vector ω_k and the scalar curvature R has the following properties:*

$$2g^{ks}[\nabla_s(2R_{ik} + R_{ki})] = 3(\partial_i R + 2R\omega_i), \tag{35}$$

where $\partial_i R = \frac{\partial R}{\partial x^i}$.

Proof. Since the Weyl connection ∇ is symmetric, then the second Bianchi identity holds, i.e. $\nabla_m R^s_{ijk} + \nabla_i R^s_{jmk} + \nabla_j R^s_{mik} = 0$. By contracting of index m and s, according to the first Bianchi identity, it follows:

$$\nabla_i R_{jk} - \nabla_j R_{ik} = \nabla_s R^s_{ijk}. \tag{36}$$

Using (2) and (14), we find the covariant derivative of R^s_{ijk}. Then after contracting on (36) by g^{jk}, we obtain (35). □

There is known [6], the Weyl space W_3 is Riemannian V_3 if and only if $\nabla_i\omega_j = \nabla_j\omega_i$. In this case, according to (15), the Ricci tensor is symmetric. If the Ricci tensor on W_3 is skew-symmetric, then W_3 is not Riemannian. We consider a 3-dimensional Weyl space when the Ricci tensor is skew-symmetric, i.e.

$$R_{jk} = -R_{kj}. \tag{37}$$

Theorem 6. *Let W_3 be a 3-dimensional Weyl space with skew-symmetric Ricci tensor. Then the following relations hold for the curvature tensor and the Ricci tensor:*

$$2R_{jk} = 3(\nabla_j\omega_k - \nabla_k\omega_j), \tag{38}$$

$$R^s_{ijk} = \frac{1}{3}\{(g_{jk}R_{il} - g_{ik}R_{jl})g^{ls} + R_{jk}\delta^s_i - R_{ik}\delta^s_j + 2R_{ji}\delta^s_k\}, \tag{39}$$

$$g^{sk}\nabla_s R_{ik} = 0. \tag{40}$$

Proof. The equality (15) and (37) imply (38). From (37) we receive: $R = g^{jk}R_{jk} = -g^{jk}R_{kj} = 0$ and $\partial_i R = 0$. Then, because $S_{jk} = R_{jk}$, we obtain (39). In this case equation (35) has the form (40). □

Theorem 7. *Let W_3 be a 3-dimensional Weyl space with skew-symmetric Ricci tensor. Then the following relations hold for the Ricci tensor:*

$$\nabla_i R_{jk} + \nabla_j R_{ki} + \nabla_k R_{ij} = 0. \tag{41}$$

Proof. Using (38), we find the covariant derivative of R_{jk}, i.e.

$$\frac{2}{3}\nabla_i R_{jk} = \nabla_i\nabla_j\omega_k - \nabla_i\nabla_k\omega_j. \tag{42}$$

From (42) we obtain the cyclic sum with respect to i, j, k:

$$\begin{aligned}\frac{2}{3}(\nabla_i R_{jk} + \nabla_j R_{ki} + \nabla_k R_{ij}) &= \\ = \nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k + \nabla_k\nabla_i\omega_j - \nabla_i\nabla_k\omega_j &+ \nabla_j\nabla_k\omega_i - \nabla_k\nabla_j\omega_i.\end{aligned} \tag{43}$$

There is known [6], the integrablity conditions for the covector ω_k have the form:

$$\nabla_i\nabla_j\omega_k - \nabla_j\nabla_i\omega_k = -R^s_{ijk}\omega_s. \tag{44}$$

Using (44) in the right side of (43), we obtain:

$$\frac{2}{3}(\nabla_i R_{jk} + \nabla_j R_{ki} + \nabla_k R_{ij}) = -(R^s_{ijk} + R^s_{kij} + R^s_{jki})\omega_s.$$

According to the first Bianchi identity, it follows (41). □

Theorem 8. *Let the composition $W_3(X_2 \times X_1)$ be geodesic-Chebyshevian and the curves $\underset{1}{v}$ and $\underset{2}{v}$ be geodesic. If the covector $\overset{3}{\underset{3}{T}}_k$ from (26) is a gradient, then the following relations hold for the Ricci tensor on W_3:*

$$R_{jk} - R_{kj} = 3e\left[\overset{3}{v}_j\left(c\overset{1}{v}_k + d\overset{2}{v}_k\right) - \overset{3}{v}_k\left(c\overset{1}{v}_j + d\overset{2}{v}_j\right)\right],\ c, d, e \in \mathbb{R} \tag{45}$$

Proof. Using the second equality of (28), after covariant differentiation of (26), we have:

$$\nabla_j \overset{3}{\underset{3}{T}}_k = \nabla_j \omega_k - e\overset{3}{v}_j\left(c\overset{1}{v}_k + d\overset{2}{v}_k + e\overset{3}{v}_k\right).$$

Since $\nabla_j \overset{3}{\underset{3}{T}}_k = \nabla_k \overset{3}{\underset{3}{T}}_j$, then using (15) and the alternation of the last equation, we find (45). □

In the case when the Ricci tensor is skew-symmetric (45) imply:

$$2R_{jk} = 3e\left[\overset{3}{v}_j\left(c\overset{1}{v}_k + d\overset{2}{v}_k\right) - \overset{3}{v}_k\left(c\overset{1}{v}_j + d\overset{2}{v}_j\right)\right]. \tag{46}$$

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ВЪРХУ НЯКОИ СПЕЦИАЛНИ КОМПОЗИЦИИ И КРИВИННИ СВОЙСТВА НА ТРИМЕРНО ВАЙЛОВО ПРОСТРАНСТВО

Добринка Костадинова Грибачева

Резюме. Специални композиции, породени от мрежа в пространство със симетрична линейна свързаност се изучават в [2], [3] и [5]. В тази работа с помощта на продълженото ковариантно диференциране се характеризират специални композиции, породени от мрежа в тримерно Вайлово пространство. Намерени са уравнения за тензора на кривина и тензора на Ричи на тримерно Вайлово пространство и са дадени някои приложения.