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# PROLONGED COVARIANT DIFFERENTIATION IN AFFINELY CONNECTED SPACES

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**Abstract.** With the help of $(n+1)$ pseudovectors a prolonged covariant differentiation in an affinely connected space without a torsion $A_n$ is introduced. It is proved that the prolonged covariant differentiation preserves the law for the parallelly translation of the fields of directions along lines and that this law does not depend on the choice of the normaliser. Derivative equations for the fields of directions are written, relations between their coefficients are found and applications are made.

The characteristics by the normaliser and the coefficients for equiaffine spaces are obtained.

Affinely connected spaces without a torsion in which there exist n compositions of base manyfolds $X_{n-1}$ and $X_1$ are studied. Characteristics of these spaces when the compositions are geodesic, chebishevian or cartesian are found.

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**Key words:** affinely connected space, net, prolonged covariant differentiation, composition, chebyshevian, cartesian, geodesic compositions

## 1. Introduction

Generalizing the notion net in two dimentional space $X_2$ Norden reachs to the notion composition [4]. Norden and Timofeev introduce the special compositions in affinely connected spaces $A_2$ [2]. The prolonged covariant differentiation, introduced in $A_2$ [3], in $W_n$ [5] and in $A_{2n}$ [6], reduce the difficulties in the investigations of the pseudoquantities. The purpose of the present paper is the introduction of the prolonged covariant differentiation in $n-$dimensional affinelly connected spaces $A_n$ without a torsion. The choice of the covector fields $\overset{\alpha}{v}_i$ corresponding to the fields of directions $\underset{\alpha}{v}^i$ as well as the new choice of the normalizer allow considerations in arbitrary dimensional affinelly connected spaces without a torsion.

The connection between nets and compositions in $A_n$ is shown in [8]. There the composition is defined with the help of an affinor connected with a net. Using this affinor it is made an application of the introduced prolonged covariant differentiation in the present paper.

## 2. Preliminary

Let the pseudo-vectors $\underset{\alpha}{v}^i(\alpha = 1, 2, \ldots, n{+}1)$, satisfying the condition [6], [7]

$$\sum_{\alpha=1}^{n+1} \underset{\alpha}{v}^i = 0 \tag{1}$$

be given in the affinely connected space without a torsion $A_n$.

We suppose that any $n$ pseudo-vectors from $\underset{\alpha}{v}^i(\alpha = 1, 2, \ldots, n{+}1)$ are linearly independent. From (1) it follows that the renormalization of the pseudo-vectors $\underset{\alpha}{v}^i(\alpha = 1, 2, \ldots, n + 1)$ is defined to with in a common non-zero factor $\sigma$ where $\sigma$ is a function of the point.

The covector fields $\overset{\alpha}{v}_i$ are defined by the conditions

$$\overset{\alpha}{v}_i \underset{\beta}{v}^i = \delta^\alpha_\beta \quad \Longleftrightarrow \quad \overset{\alpha}{v}_i \underset{\alpha}{v}^s = \delta^s_i \ , \ (\alpha, \beta = 1, 2, \ldots, n), \tag{2}$$

$$\overset{n+1}{v}_i = -\sum_{\alpha=1}^{n} \overset{\alpha}{v}_i. \tag{3}$$

From (1), (2), (3) it follows

$$\overset{n+1}{v}_i \underset{\alpha}{v}^i = -1, \quad \underset{n+1}{v}^i \overset{\alpha}{v}_i = -1, \quad (\alpha = 1, 2, \ldots, n), \quad \underset{n+1}{v}^i \overset{n+1}{v}_i = n. \tag{4}$$

According to [1] the field of directions $v^i$ is parallelly translating along the lines $(w)$ if and only if

$$\nabla_k v^i w^k = \lambda v^i, \tag{5}$$

where $\lambda$ is an arbitrary function. We have noticed with $\nabla$ the covariant derivative defined by the coefficients of the connectedness $\Gamma^k_{is}$ of the space $A_n$.

Pseudo-quantities $A$ which after a renormalization of $\underset{\alpha}{v}^i$ are transformed by the law $\breve{A} = \sigma^k A$ are called satellites of $\underset{\alpha}{v}^i$ of weight $\{k\}$ [3] .

From (2) it follows that $\overset{\alpha}{v}_i$ $(\alpha = 1, 2, \ldots, n)$ are satellites of the $\underset{\alpha}{v}^i$ of weight $\{-1\}$, i.e. $\breve{\overset{\alpha}{v}}_i = \sigma^{-1}\overset{\alpha}{v}_i$.

A normalizer is called any covector admitting a transformation of the form [3]

$$\breve{T}_i = T_i + \partial_i ln\lambda \ . \tag{6}$$

According to [3] a prolonged covariant derivative of pseudoquantities with weight $\{k\}$ is called the object

$$\overset{\bullet}{\nabla}_s A = \nabla_s A - kT_s A \ . \tag{7}$$

Let notice by $(\underset{\alpha}{v})$ the lines determined from the pseudo-vectors $\underset{\alpha}{v}^i$, $(\alpha{=}1, 2, \ldots, n)$ and by $(\underset{1}{v}, \underset{2}{v}, \ldots \underset{n}{v})$ the net determined from the pseudo-vectors $\underset{\alpha}{v}^i$, $(\alpha{=}1, 2, \ldots, n)$ and by $(\underset{1}{v}, \underset{2}{v}, \ldots \underset{n+1}{v})$ the $n{+}1$-web determined from the pseudo-vectors $\underset{\alpha}{v}^i$, $(\alpha = 1, 2, \ldots, n+1)$.

The following affinor

$$a^{\beta}_{\alpha} = \sum_{i=1}^{m} \underset{i}{v}^{\beta} \overset{i}{v}_{\alpha} - \sum_{i=m+1}^{n} \underset{i}{v}^{\beta} \overset{i}{v}_{\alpha} \tag{8}$$

uniquely determinated from the net$(\underset{1}{v}, \underset{2}{v}, \ldots \underset{n}{v})$ is introduced in [8]. Since $a^{\beta}_{\alpha}\, a^{\sigma}_{\beta} = \delta^{\sigma}_{\alpha}$ , according to [4] the affinor (8) defines a composition $X_m \times X_{n-m}$ in $A_n$.

According to [2], [4] the following definitions can be write:

The composition $(X_m \times X_{n-m}) \in A_n$ is called cartesian if the positions $P(X_m)$ and $P(X_{n-m})$ are parallelly translated along any line in the space .

The composition $(X_m \times X_{n-m}) \in A_n$ is called geodesic if the positions $P(X_m)$ and $P(X_{n-m})$ are parallelly translated along any line of $X_m$ and $X_{n-m}$ respectively .

The composition $(X_m \times X_{n-m}) \in A_n$ is called chebyshevian if the positions $P(X_m)$ and $P(X_{n-m})$ are parallelly translated along any line of $X_{n-m}$ and $X_m$ respectively.

### 3. Prolonged covariant differentiation in $A_n$

Consider the covector

$$(9) \qquad T_k = \frac{1}{n} \overset{n+1}{v}_s \nabla_k \underset{n+1}{v}^s .$$

After renormalization of the pseudo-vectors $\underset{\alpha}{v}^i$, $(\alpha = 1, 2, \dots, n+1)$ for $T_k$ we can write

$$\breve{T}_k = \frac{1}{n}\sigma^{-1}\overset{n+1}{v}_s \nabla_k (\sigma \underset{n+1}{v}^s) = \frac{1}{n}\sigma^{-1}\sigma_k \overset{n+1}{v}_s \underset{n+1}{v}^s + \frac{1}{n}\sigma^{-1}\sigma \overset{n+1}{v}_s \nabla_k \underset{n+1}{v}^s,$$

from where taking into account (2), (4), (9) we find $\breve{T}_k = \partial_k ln\sigma + T_k$. Then we can choose the covector (9) as a normalizer [3]. The existence of this normalizer allows us to introduce the prolonged covariant differentiation of the satellites of the pseudo-vectors $\underset{\alpha}{v}^i$ with weight $\{k\}$ by the formula (7).

**Lemma 1.** *The pseudo-vector $v^i$ is parallelly translated along the lines $(w)$ if and only if $\overset{\bullet}{\nabla}_s v^i \; w^s = \lambda v^i$ , where $\lambda$ is an arbitrary function. This law of the parallelly translating does not depend on the choice of the normalizer.*

**Proof.** Let $v^i$ has a weight $\{k\}$ and let notice by ${}^1\overset{\bullet}{\nabla}_s$ the prolonged covariant differentiation introduced with the help of an arbitrary normalizer $Q_k$ which is different from $T_k$. According to (7) we have $\overset{\bullet}{\nabla}_s v^i = \nabla_s v^i - kT_s v^i$ , ${}^1\overset{\bullet}{\nabla}_s v^i = \nabla_s v^i - kQ_s v^i$. If we accept the notations $T_s w^s = \mu$, $Q_s w^s = \nu$, we obtain $\overset{\bullet}{\nabla}_s v^i w^s = \nabla_s v^i w^s - k\mu v^i$, ${}^1\overset{\bullet}{\nabla}_s v^i w^s = \nabla_s v^i w^s - k\nu v^i$. Now it is easy to see that the equalities $\nabla_s v^i \; w^s = \lambda v^i$ , $\overset{\bullet}{\nabla}_s v^i \; w^s = \rho v^i$, ${}^1\overset{\bullet}{\nabla}_s v^i \; w^s = \tau v^i$ , where $\lambda, \rho, \tau$ are arbitrary functions, are equivalent. □

### 4. Derivative equations

The prolonged covariant derivative of the field of directions $\underset{\alpha}{v}^i$, $(\alpha = 1, 2, \dots, n)$ can be presented in the following way

$$(10) \qquad \overset{\bullet}{\nabla}_k \underset{\alpha}{v}^i = \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}^i \; , \; (\alpha = 1, 2, \dots, n) \; ,$$

because $\underset{\alpha}{v}^i$, $(\alpha = 1, 2, \dots, n)$ are independent pseudo-vectors.

From (2) and (10) it follows

$$\overset{\bullet}{\nabla}_k \overset{\alpha}{v}_i = -\underset{\sigma}{\overset{\alpha}{T}}_k \overset{\sigma}{v}_i \ , \ (\alpha = 1, 2, \dots, n) \ . \tag{11}$$

The equalities (10) and (11) are called derivative equations. Obviously the coefficients $\underset{\sigma}{\overset{\alpha}{T}}_k$ from the derivative equations have weights $\{0\}$.

With the help of (1), (3) , (10) and (11) we find

$$\overset{\bullet}{\nabla}_s \underset{n+1}{v}{}^i = -\sum_{\alpha=1}^{n} \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}{}^i \ \ , \ \ \overset{\bullet}{\nabla}_s \overset{n+1}{v}{}_i = \sum_{\alpha=1}^{n} \underset{\sigma}{\overset{\alpha}{T}}_k \overset{\sigma}{v}_i \ . \tag{12}$$

**Lemma 2.** *The coefficients* $\underset{\sigma}{\overset{\alpha}{T}}_k$ *from the derivative equations satisfy the following condition* $\sum\limits_{\alpha=1}^{n} \sum\limits_{\beta=1}^{n} \underset{\beta}{\overset{\alpha}{T}}_k = 0$ .

**Proof.** According to (7) and (12) we have $\overset{\bullet}{\nabla}_s \underset{n+1}{v}{}^i = -\sum\limits_{\alpha=1}^{n} \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}{}^i = \nabla_k \underset{n+1}{v}{}^i - T_k \underset{n+1}{v}{}^i$, from where after contraction by $\overset{n+1}{v}{}_i$ and taking into account (4) and (9) we establish the validity of Lemma 2. □

**Theorem 1.** *The affinely connected space* $A_n$ *is equiaffine if and only if the normalizer* $T_k$ *and the coefficients* $\underset{\alpha}{\overset{\sigma}{T}}_k$ *from the derivative equations satisfy the equality* $n\nabla_{[s}T_{k]} + \nabla_{[s}\underset{\alpha}{\overset{\alpha}{T}}_{k]} = 0$.

**Proof.** After the covariant differentiation of (10) and taking into account the equalities $\overset{\bullet}{\nabla}_s \underset{\alpha}{v}{}^i = \nabla_k \underset{\alpha}{v}{}^i - T_k \underset{\alpha}{v}{}^i = \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}{}^i$ we obtain $\nabla_s \nabla_k \underset{\alpha}{v}{}^i - \nabla_s T_k \underset{\alpha}{v}{}^i - T_k(T_s \underset{\alpha}{v}{}^i + \underset{\alpha}{\overset{\sigma}{T}}_s \underset{\sigma}{v}{}^i) = \nabla_s \underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}{}^i + \underset{\alpha}{\overset{\sigma}{T}}_k (T_s \underset{\sigma}{v}{}^i + \underset{\sigma}{\overset{\beta}{T}}_s \underset{\beta}{v}{}^i)$. Now let apply the integrability condition. So we find $\frac{1}{2} R_{skm.}{}^{i} \underset{\alpha}{v}{}^m = \nabla_{[s}T_{k]} \underset{\alpha}{v}{}^i + \nabla_{[s}\underset{\alpha}{\overset{\sigma}{T}}_{k]} \underset{\sigma}{v}{}^i + \underset{\alpha}{\overset{\sigma}{T}}_{[k} \underset{\sigma}{\overset{\beta}{T}}_{s]} \underset{\beta}{v}{}^i$, where $R_{skm.}{}^{i}$ is the tensor of the curvature of the space $A_n$.

According to (2) after contraction with $\overset{\alpha}{v}_j$ we find $\frac{1}{2} R_{skj.}{}^{i} = \nabla_{[s}T_{k]} \delta^i_j + \nabla_{[s}\underset{\alpha}{\overset{\sigma}{T}}_{k]} \underset{\sigma}{v}{}^i \overset{\alpha}{v}_j + \underset{\alpha}{\overset{\sigma}{T}}_{[k} \underset{\sigma}{\overset{\beta}{T}}_{s]} \underset{\beta}{v}{}^i \overset{\alpha}{v}_j$, from where it follows $\frac{1}{2} R_{ski.}{}^{i} = n\nabla_{[s}T_{k]} + \nabla_{[s}\underset{\alpha}{\overset{\alpha}{T}}_{k]}$. But it is known that the affinely connected space $A_n$ is equiaffine if and only if $R_{ski.}^{\ \ \ i} = 0$. So we obtain the affinely connected space $A_n$ is equiaffine if and only if $n\nabla_{[s}T_{k]} + \nabla_{[s}\overset{\alpha}{\underset{\alpha}{T}}_{k]} = 0$ $\square$

## 5. Application of the prolonged covariant differentiation

Consider $n$ affinors (8) defined by the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in A_n$ [8] when $m = n-1$.

$$
\begin{aligned}
\underset{1}{a}{}_i^s &= -\underset{1}{v}{}^s \overset{1}{v}_i + \underset{2}{v}{}^s \overset{2}{v}_i + \underset{3}{v}{}^s \overset{3}{v}_i + \cdots + \underset{n}{v}{}^s \overset{n}{v}_i, \\
\underset{2}{a}{}_i^s &= \underset{1}{v}{}^s \overset{1}{v}_i - \underset{2}{v}{}^s \overset{2}{v}_i + \underset{3}{v}{}^s \overset{3}{v}_i + \cdots + \underset{n}{v}{}^s \overset{n}{v}_i, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\underset{n}{a}{}_i^s &= \underset{1}{v}{}^s \overset{1}{v}_i + \underset{2}{v}{}^s \overset{2}{v}_i + \underset{3}{v}{}^s \overset{3}{v}_i + \cdots - \underset{n}{v}{}^s \overset{n}{v}_i.
\end{aligned}
\tag{13}
$$

Because of $\underset{\alpha}{a}{}_i^s \underset{\alpha}{a}{}_s^k = \delta_i^k$, $(\alpha = 1, 2, ..., n)$, [2] and [4] it follows that the affinely connected space $A_n$ with the affinors $\underset{\alpha}{a}{}_i^s$ $(\alpha = 1, 2, ..., n)$ is a space of compositions of two base manifolds $X_{n-1}$ and $X_1$ . For any $\alpha = 1, 2, ..., n$ two positions $P(X_{n-1})$ and $P(X_1)$ of the base manifolds $X_{n-1}$ and $X_1$ respectively pass through any point of the space $A_n$ [2], [4].

Here using the definitions about special compositions [2], [4] we can write:

The composition $(X_{n-1} \times X_1) \in A_n$ is called cartesian if the positions $P(X_{n-1})$ and $P(X_1)$ are parallelly translated along any line in the space .

The composition $(X_{n-1} \times X_1) \in A_n$ is called geodesic if the positions $P(X_{n-1})$ and $P(X_1)$ are parallelly translated along any line of $X_{n-1}$ and $X_1$ respectively .

The composition $(X_{n-1} \times X_1) \in A_n$ is called chebyshevian if the positions $P(X_{n-1})$ and $P(X_1)$ are parallelly translated along any line of $X_1$ and $X_{n-1}$ respectively .

When all affinors (13) define in $A_n$ cartesian or geodesic or chebyshevian compositions then the space $A_n$ will be notice ${}^cA_n$ or ${}^gA_n$ or ${}^{ch}A_n$, respectively.

Since the affinors (13) have weghts $\{0\}$ then

$$
\nabla_k \underset{\alpha}{a}{}_i^s = \overset{\bullet}{\nabla}_k \underset{\alpha}{a}{}_i^s\ , \qquad \alpha = 1, 2, ..., n. \tag{14}
$$

### 5.1. Geodesic compositions in $A_n$

**Theorem 2.** *The space $A_n$ is the space ${}^gA_n$ if and only if*

$$\underset{\alpha}{\overset{\beta}{T}}_i = \underset{\beta}{\overset{\beta}{\lambda}}\, \overset{\beta}{v}_i \quad , \quad \alpha, \beta = 1, 2, ..., n; \alpha \neq \beta \ , \tag{15}$$

*where $\underset{\beta}{\overset{\beta}{\lambda}}$ are functions of the point.*

**Proof.** Let the affinors (13) define geodesic compositions in $A_n$. According to [2] the affinors $\underset{\alpha}{a}{}_i^s$ define geodesic compositions in $A_n$, if and only if $\underset{\alpha}{a}{}_i^s \nabla_k \underset{\alpha}{a}{}_s^j + \underset{\alpha}{a}{}_k^s \nabla_s \underset{\alpha}{a}{}_i^j = 0$. Because of (14) the last equality accepts the form

$$\underset{\alpha}{a}{}_i^s \overset{\bullet}{\nabla}_k \underset{\alpha}{a}{}_s^j + \underset{\alpha}{a}{}_k^s \overset{\bullet}{\nabla}_s \underset{\alpha}{a}{}_i^j = 0 \tag{16}$$

From (10), (11), (13) it follows that (16) can be written in the form
$\underset{\alpha}{a}{}_i^s \overset{\bullet}{\nabla}_k \underset{\alpha}{a}{}_s^j + \underset{\alpha}{a}{}_k^s \overset{\bullet}{\nabla}_s \underset{\alpha}{a}{}_i^j = 2(\underset{1}{\overset{\alpha}{T}}_k + \underset{\alpha}{a}{}_k^s \underset{1}{\overset{\alpha}{T}}_s)\, \underset{\alpha}{v}{}^j \overset{1}{v}_i + \cdots +$
$2(\underset{\alpha-1}{\overset{\alpha}{T}}_k + \underset{\alpha}{a}{}_k^s \underset{\alpha-1}{\overset{\alpha}{T}}_s)\, \underset{\alpha}{v}{}^j \overset{\alpha-1}{v}_i + 2(\underset{\alpha+1}{\overset{\alpha}{T}}_k + \underset{\alpha}{a}{}_k^s \underset{\alpha+1}{\overset{\alpha}{T}}_s)\, \underset{\alpha}{v}{}^j \overset{\alpha+1}{v}_i + \cdots +$
$2(\underset{n}{\overset{\alpha}{T}}_k + \underset{\alpha}{a}{}_k^s \underset{n}{\overset{\alpha}{T}}_s)\, \underset{\alpha}{v}{}^j \overset{n}{v}_i + 2[(\underset{\alpha}{\overset{1}{T}}_k - \underset{\alpha}{a}{}_k^s \underset{\alpha}{\overset{1}{T}}_s)\, \underset{1}{v}{}^j + \cdots + (\underset{\alpha}{\overset{\alpha-1}{T}}_k - \underset{\alpha}{a}{}_k^s \underset{\alpha}{\overset{\alpha-1}{T}}_s)\, \underset{\alpha-1}{v}{}^j + \cdots +$
$(\underset{\alpha}{\overset{\alpha+1}{T}}_k - \underset{\alpha}{a}{}_k^s \underset{\alpha}{\overset{\alpha+1}{T}}_s)\, \underset{\alpha+1}{v}{}^j + \cdots + (\underset{\alpha}{\overset{n}{T}}_k - \underset{\alpha}{a}{}_k^s \underset{\alpha}{\overset{n}{T}}_s)\, \underset{n}{v}{}^j] \overset{\alpha}{v}_i = 0.$

From the independence of the fields of directions $\underset{\alpha}{v}{}^i$ and $\overset{\alpha}{v}_i$ $(\alpha = 1, 2, \dots n)$ it follows that the equality will be fulfilled if and only if $\underset{\beta}{\overset{\alpha}{T}}_k + \underset{\alpha}{a}{}_k^s \underset{\beta}{\overset{\alpha}{T}}_s = 0$ and $\underset{\alpha}{\overset{\beta}{T}}_k - \underset{\alpha}{a}{}_k^s \underset{\alpha}{\overset{\beta}{T}}_s = 0$. But $\underset{\alpha}{\overset{\beta}{T}}_k - \underset{\alpha}{a}{}_k^s \underset{\alpha}{\overset{\beta}{T}}_s = 0 \Leftrightarrow (\underset{\alpha}{\overset{\beta}{T}}_k - \underset{\alpha}{a}{}_k^s \underset{\alpha}{\overset{\beta}{T}}_s) \underset{\alpha}{v}{}^k = 0$, from where taking into account (2) and (13) we obtain $\underset{\alpha}{\overset{\beta}{T}}_k \underset{\alpha}{v}{}^k = 0$. Thus we find the following presentation of the coefficients of the derivative equations

$$\underset{\alpha}{\overset{\beta}{T}}_k = \sum_{i=1}^{\alpha-1} \underset{i}{\overset{\beta}{\lambda}}\, \overset{i}{v}_k \ + \sum_{i=\alpha+1}^{n} \underset{i}{\overset{\beta}{\lambda}}\, \overset{i}{v}_k \ . \tag{17}$$

Now after substitution of $\underset{\alpha}{\overset{\beta}{T}}_k$ from (17) and $\underset{\alpha}{a}{}_k^s$ from (13) in $\underset{\beta}{\overset{\alpha}{T}}_k + \underset{\alpha}{a}{}_k^s \underset{\beta}{\overset{\alpha}{T}}_s = 0$ we establish $\underset{\alpha}{\overset{\beta}{T}}_k - \underset{\beta}{\overset{\beta}{\lambda}} \overset{\beta}{v}_k = 0$, which means that $\underset{\alpha}{\overset{\beta}{\lambda}} = o$ for any $\alpha \neq \beta$. □

**Theorem 3.** *If the net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v}) \in A_n$ *is chosen as a coordinate one, then the space* $A_n$ *is the space* ${}^gA_n$ *if and only if* $\Gamma^k_{is}=0$ *for any* $k\neq i, k\neq s$.

**Proof.** Let the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v}) \in A_n$ be chosen as a coordinate one. According to (2) the conditions (15) are equivalent to the following conditions

$$\underset{\alpha}{\overset{\beta}{T}}{}_i \underset{\sigma}{v}{}^i = 0 \quad \text{for} \quad \text{any} \quad \alpha\neq\beta,\beta\neq\sigma. \tag{18}$$

With the help of (2), (7), (10) we obtain the following presentation for the coefficients from the derivative equations

$$\underset{\alpha}{\overset{\sigma}{T}}{}_i = \overset{\sigma}{v}_s(\partial_i \underset{\alpha}{v}{}^s + \Gamma^s_{ip}\underset{\alpha}{v}{}^p - T_i \underset{\alpha}{v}{}^s). \tag{19}$$

Because of (2) , (18) and (19) for the coefficients of the connectedness and the fields of directions of the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v}) \in A_n$ we find $\Gamma^s_{ip}\ \underset{\alpha}{v}{}^p\ \overset{\sigma}{v}_s\ \underset{\beta}{v}{}^i=0$, for any $\alpha\neq\beta,\beta\neq\sigma$. Since the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{n}{v}) \in A_n$ is coordinate then the last equality is fulfilled if and only if $\Gamma^k_{is}=0$ for any $k\neq i, k\neq s$. □

### 5.2. Chebyshevian compositions in $A_n$

**Theorem 4.** *The space* $A_n$ *is the space* ${}^{ch}A_n$ *if and only if*

$$\underset{\alpha}{\overset{\beta}{T}}{}_i = \overset{\beta}{\lambda}\overset{\alpha}{v}_i \quad , \quad \alpha,\beta=1,2,...,n;\alpha\neq\beta \ , \tag{20}$$

*where* $\overset{\beta}{\lambda}$ *are functions of the point.*

**Proof.** Let the affinors $\underset{\alpha}{a}{}^s_i$ $(\alpha=1,2,...,n)$ define chebyshevian compositions in $A_n$. According to [2] the affinors $\underset{\alpha}{a}{}^s_i$ define chebyshevian compositions in $A_n$, if and only if $\nabla_{[i}\underset{\alpha}{a}{}^{\ k}_{s]}=0$. Because of (14) the last equality accepts the form

$$\overset{\bullet}{\nabla}_{[i}\ \underset{\alpha}{a}{}_{s]}{}^k=0. \tag{21}$$

From (10), (11), (13) it follows

$$\overset{\bullet}{\nabla}_{[i}\ \underset{\alpha}{a}{}^k_{s]} = -2\underset{\alpha}{\overset{1}{T}}{}_{[i}\overset{\alpha}{v}_{s]}\underset{1}{v}{}^k - 2\underset{\alpha}{\overset{2}{T}}{}_{[i}\overset{\alpha}{v}_{s]}\underset{2}{v}{}^k - \dots - 2\underset{\alpha}{\overset{\alpha-1}{T}}{}_{[i}\overset{\alpha}{v}_{s]}\underset{\alpha-1}{v}{}^k - 2\underset{\alpha}{\overset{\alpha+1}{T}}{}_{[i}\overset{\alpha}{v}_{s]}\underset{\alpha+1}{v}{}^k - \dots$$
$$-2\underset{\alpha}{\overset{n}{T}}{}_{[i}\overset{\alpha}{v}_{s]}\underset{n}{v}{}^k + 2(\underset{\alpha}{\overset{1}{T}}{}_{[i}\overset{\alpha}{v}_{s]} + \underset{\alpha}{\overset{2}{T}}{}_{[i}\overset{\alpha}{v}_{s]} + \dots + \underset{\alpha}{\overset{\alpha-1}{T}}{}_{[i}\overset{\alpha}{v}_{s]} + \underset{\alpha}{\overset{\alpha+1}{T}}{}_{[i}\overset{\alpha}{v}_{s]} + \dots + \underset{\alpha}{\overset{n}{T}}{}_{[i}\overset{\alpha}{v}_{s]})\underset{\alpha}{v}{}^k$$

From the independence of the fields of directions $\underset{\alpha}{v}^i$ and $\overset{\alpha}{v}_i$ $(\alpha = 1, 2, \dots n)$ it follows that the right-hand side of the last equality is equal to zero if and only if $\overset{\beta}{\underset{\alpha}{T}}_{[i}\overset{\alpha}{v}_{s]} = 0$ for any $\alpha, \beta = 1, 2, \dots n; \alpha \neq \beta$. But it is obviously that the last conditions are equivalent to the conditions (20). $\square$

**Theorem 5.** *If the net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in A_n$ *is chosen as a coordinate one, then the space* $A_n$ *is the space* ${}^{ch}A_n$ *if and only if* $\Gamma^k_{is} = 0$ *for any* $i \neq s, k \neq s$.

**Proof.** According to (2) and (19) for the coefficients of the connectedness and the fields of directions of the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in A_n$ we find $\Gamma^k_{is}\, \underset{\alpha}{v}^s\, \overset{\sigma}{v}_k\, \underset{\beta}{v}^i = 0$, for any $\alpha \neq \sigma, \beta \neq \sigma$. Since the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in A_n$ is coordinate then the last equality is fulfilled if and only if $\Gamma^k_{is} = 0$ for any $i \neq s, k \neq s$. $\square$

### 5.3. Cartesian compositions in $A_n$

**Theorem 6.** *The space* $A_n$ *is the space* ${}^{c}A_n$ *if and only if*

$$\overset{\beta}{\underset{\alpha}{T}}_i = 0 \quad , \quad \alpha, \beta = 1, 2, ..., n; \alpha \neq \beta \ , \tag{22}$$

**Proof.** Let the affinors $\underset{\alpha}{a}^k_s$ $(\alpha = 1, 2, ..., n)$ define cartesian compositions in $A_n$. According to [2] the affinors $\underset{\alpha}{a}^k_s$ define cartesian compositions in $A_n$, if and only if $\nabla_i \underset{\alpha}{a}{}^k_s = 0$. Because of (14) the last equality accepts the form

$$\overset{\bullet}{\nabla}_i\ \underset{\alpha}{a_s}{}^k = 0. \tag{23}$$

From (10), (11), (13) it follows

$$\overset{\bullet}{\nabla}_i\ \underset{\alpha}{a}^k_s = -2\overset{1}{\underset{\alpha}{T}}_i\ \overset{\alpha}{v}_s\ \underset{1}{v}^k - 2\overset{2}{\underset{\alpha}{T}}_i\ \overset{\alpha}{v}_s\ \underset{2}{v}^k - \dots - 2\ \overset{\alpha-1}{\underset{\alpha}{T}}{}_i\ \overset{\alpha}{v}_s\ \underset{\alpha-1}{v}{}^k - 2\ \overset{\alpha+1}{\underset{\alpha}{T}}{}_i\ \overset{\alpha}{v}_s\ \underset{\alpha+1}{v}{}^k - \dots$$

$$-2\overset{n}{\underset{\alpha}{T}}_i\ \overset{\alpha}{v}_s\ \underset{n}{v}^k + 2(\overset{1}{\underset{\alpha}{T}}_i\ \overset{\alpha}{v}_s + \overset{2}{\underset{\alpha}{T}}_i\ \overset{\alpha}{v}_s + \dots + \overset{\alpha-1}{\underset{\alpha}{T}}{}_i\ \overset{\alpha}{v}_s + \overset{\alpha+1}{\underset{\alpha}{T}}{}_i\ \overset{\alpha}{v}_s + \dots + \overset{n}{\underset{\alpha}{T}}_i\ \overset{\alpha}{v}_s)\underset{\alpha}{v}^k.$$

From the independence of the fields of directions $\underset{\alpha}{v}^i$ and $\overset{\alpha}{v}_i$ $(\alpha = 1, 2, \dots n)$ it follows that the right-hand side of the last equality is equal to zero if and only if $\overset{\beta}{\underset{\alpha}{T}}_i\overset{\alpha}{v}_s = 0$ for any $\alpha, \beta = 1, 2, \dots n; \alpha \neq \beta$. Since $\overset{\alpha}{v}_s$ are independent, then $\overset{\beta}{\underset{\alpha}{T}}_i = 0$. $\square$

From Theorems 2, 4 and 6 it follows

**Corollary 1.** *If the affinors* $\underset{\alpha}{a}_s^k$ $(\alpha = 1, 2, ..., n)$ *define at the same time geodesic and chebyshevian compositions in* $A_n$ *then they define and cartesian ones.*

From Theorems 3, 5 and Corollary 1 it follow

**Corollary 2.** *If the net* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v}) \in A_n$ *is chosen as a coordinate one, then the space* $A_n$ *is the space* ${}^cA_n$ *if and only if* $\Gamma_{is}^k = 0$ *for any* $i \neq s,\ k \neq s,\ k \neq i$.

**Corollary 3.** *If the affinors* $\underset{\alpha}{a}_s^k$ $(\alpha = 1, 2, ..., n)$ *define cartesian compositions in* $A_n$ *then the coefficients from the derivative equations satisfy the equality* $\underset{\alpha}{\overset{\alpha}{T}}_i = 0$.

**Proof.** Let $A_n$ be the space ${}^cA_n$. According to Theorem 6 the derivative equations accept the form

$$\overset{\bullet}{\nabla}_k \underset{\alpha}{v}^i = \underset{\alpha}{\overset{\alpha}{T}}_k \underset{\alpha}{v}^i\ ,\ (\alpha = 1, 2, \dots, n). \tag{24}$$

Then using (1), (2), (3), (9), (7), (24) we find

$$T_k = \frac{1}{n} \overset{n+1}{v}{}_i \nabla_k \underset{n+1}{v}{}^i = \frac{1}{n} \overset{n+1}{v}{}_i (\overset{\bullet}{\nabla}_k \underset{n+1}{v}{}^i + T_k \underset{n+1}{v}{}^i) = \frac{1}{n} \underset{\alpha}{\overset{\alpha}{T}}_i + T_k,$$

from where it follows $\underset{\alpha}{\overset{\alpha}{T}}_i = 0$. □

From Theorem 1 and Corollary 3 it follows

**Corollary 4.** *The space* ${}^cA_n$ *is equiaffine if and only if the normaliser is gradient.*

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# ПРОДЪЛЖЕНО КОВАРИАНТНО ДИФЕРЕНЦИРАНЕ В ПРОСТРАНСТВА С АФИННА СВЪРЗАНОСТ

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**Резюме**. В пространства с афинна свързаност без торзия се въвежда продължено ковариантно диференциране с помощта на $(n+1)$ псевдовектори. Доказано е, че продълженото ковариантно диференциране запазва закона за паралелното пренасяне на полетата от направления по линии и че този закон не зависи от избора на нормализатора. Записани са деривационните уравнения за полетата от направления, намерени са връзките между техните коефициенти и са направени приложения.

Намерени са характеристики, съдържащи нормализатора и коефициентите от деривационните уравнения, за еквиафинните пространства.

Изучават се пространства с афинна свързаност, в които съществуват $n$ композиции от базови многообразия $X_{n-1}$ и $X_1$. Намерени са характеристики на тези пространства, когато композициите са геодезични, чебишеви или декартови.