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ON EQUIVALENT ANALYTIC NORMS IN ORLICZ–LORENTZ SEQUENCE SPACES [1]

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Abstract. We prove that if the generating Orlicz function M has not Δ_2–condition at zero, then the existence of an equivalent analytic norm in the orlicz–Lorentz sequence space $d_0(w, M)$ is equivalent to $d_0(w, M)$ to be isomorphically polyhedral. We show that if $\lim_{t\to 0} \frac{M(\lambda t)}{M(t)} = \infty$ for some $\lambda > 1$ then the Orlicz–Lorentz sequence space $d_0(w, M)$ is isomorphic to a polyhedral Banach space and therefore it admits an equivalent analytic norm, it is c_0–saturated and it has a separable dual. We characterize all the c_0–saturated Orlicz–Lorentz sequence spaces.

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1. Introduction

The notion of polyhedral Banach spaces was introduced in [10]. A Banach space is called polyhedral if the unit ball of each of its finite dimensional subspaces is a polyhedron, i.e. it has finitely many extreme points. A Banach space is called isomorphically polyhedral if it is isomorphic to a polyhedral Banach space. Fundamental results about polyhedral Banach spaces can be found in [3] and [5]. Isomorphically polyhedral Banach spaces are c_0 [6], the spaces $C(\alpha)$ for any ordinal α [4]. The Orlicz sequence space h_M is isomorphically polyhedral if $\lim_{t\to 0} \frac{M(\lambda t)}{M(t)} = \infty$ for some $\lambda > 1$ [11]. The Musielak–Orlicz

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sequence spaces h_Φ is isomorphically polyhedral if h_Φ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis [2].

Recall that a Banach space is c_0–saturated if every closed infinite dimensional subspace contains a subspace which is isomorphic to c_0. It is shown [4] that any separable isomorphically polyhedral Banach space is c_0–saturated and has a separable dual. Thus the Orlicz and Musielak–Orlicz sequence spaces mentioned above are c_0–saturated.

Using the ideas of [11] we find a sufficient condition for the Orlicz–Lorentz sequence spaces $d_0(w, M)$ to be isomorphically polyhedral and we characterize all the c_0–saturated Orlicz–Lorentz sequence spaces.

It is well known that any separable, isomorphically polyhedral Banach space admits an equivalent analytic norm [1]. A general result for a Banach space with an equivalent analytic norm to be isomorphically polyhedral is obtained in [8]. This result is applied in the same article to show that for a wide class of Orlicz spaces h_M the existence of an equivalent analytic norm is equivalent to h_M to be isomorphically polyhedral. It turns out that this general result can be applied in Orlicz–Lorentz sequence spaces to investigate the same problem as well.

2. Preliminaries

A standard Banach space terminology can be found in [12].

Let us recall that an Orlicz function M is an even, continuous, nondecreasing, convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$.

The Orlicz function M is said to have the property Δ_2 at zero if there is a constant $C > 0$ such that $M(2t) \leq CM(t)$ for every $t \in [0, t_0]$ for some $t_0 > 0$ and we write $M \in \Delta_2(0)$.

To every Orlicz function M the following numbers are associated ([12], p.143):

$$\alpha_M = \sup\left\{p > 0 : \sup_{0<u,v\leq 1} \frac{M(uv)}{M(u)v^p} < \infty\right\},$$

$$\beta_M = \inf\left\{q > 0 : \inf_{0<u,v\leq 1} \frac{M(uv)}{M(u)v^q} > 0\right\}.$$

It is easy to see that $1 \leq \alpha_M \leq \beta_M \leq \infty$. A well known fact is that $M \in \Delta_2(0)$ iff $\beta_M < \infty$.

For a given Orlicz function M and $a \in (0, +\infty)$, let M_a be the function M scaled at a, defined by $M_a = \dfrac{M(at)}{M(a)}$. The following sets of functions mapping $[0, +\infty)$ into $[0, +\infty)$

$$E^0_{M,A} = \{M_a : 0 < a < A\}, \quad C_{M,A} = \overline{\mathrm{conv} E^0_{M,A}}, \quad C_M = \cap_{A>0} C_{M,A}$$

will be needed in the sequel [9], [12].

The Orlicz sequence space ℓ_M, generated by an Orlicz function M is the set of all real sequences $x = \{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty M(\lambda x_i) < \infty$ for some $\lambda > 0$. The Luxemburg norm is defined by

$$\|x\|_M = \inf \left\{ \lambda > 0 : \sum_{i=1}^\infty M\left(\frac{x_i}{\lambda}\right) \le 1 \right\}.$$

We denote by h_M the closed linear subspace of ℓ_M, generated by all $x \in \ell_M$, such that $\sum_{i=1}^\infty M(\lambda x_i) < \infty$ for every $\lambda > 0$. If $M(t) = t^p$, $p \ge 1$ we get the space ℓ_p.

An extensive study of Orlicz spaces can be found in [12].

Let $w = \{w_i\}_{i=1}^\infty$ be a positive decreasing sequence such that $w_1 = 1$, $\lim_{i\to\infty} w_i = 0$ and $\lim_{n\to\infty} W(n) = \infty$, where $W(n) = \sum_{i=1}^n w_i$ for every $n \in \mathbb{N}$. The Orlicz–Lorentz sequence space $d(w, M)$ consists of all bounded real sequences $x = \{x_i\}_{i=1}^\infty$ such that for some $\lambda > 0$ holds $I(\lambda x) < \infty$, where

$$I(x) = \sum_{i=1}^\infty w_i M(x_i^*) = \sup \left\{ \sum_{i=1}^\infty w_i M(x_{\pi(i)}) : \pi \text{ is an injection } \mathbb{N} \to \mathbb{N} \right\},$$

and $x^* = \{x_i^*\}_{i=1}^\infty$ is the decreasing rearrangement of $|x| = \{|x_n|\}_{n=1}^\infty$. The space $d(w, M)$ equipped with the Luxemburg norm

$$\|x\|_{d(w,M)} = \inf\{\lambda > 0 : I(x/\lambda) \le 1\} \tag{1}$$

is a Banach space [9].

Notice that the assumption $\lim_{n\to\infty} W(n) = \infty$ yields that $d(w, M) \hookrightarrow c_0$, where by $Y \hookrightarrow X$ we will denote that Y is isomorphic to a subspace of X.

We denote by $d_0(w, M)$ the closure of all finitely supported sequences in $d(w, M)$.

The next proposition from [9] shows that the space $d(w, M)$ has much in common with ℓ_M.

Proposition 2.1. *([9]) I) The subspace $d_0(w,M)$ coincides with the set of all sequences $x=\{x_i\}_{i=1}^\infty$ such that for every $\lambda>0$ holds $I(\lambda x)<\infty$. Moreover, the sequence of the unit vectors $\{e_i\}_{i=1}^\infty$ is a symmetric basis in $d_0(w,M)$.*
II) The following assertions are equivalent:
i) The Orlicz function M satisfies the Δ_2–condition;
ii) the unit vectors $\{e_i\}_{i=1}^\infty$ form a boundedly complete basis in $d_0(w,M)$;
iii) $d_0(w,M)=d(w,M)$;
iv) $d_0(w,M)$ does not contain a closed subspace isomorphic to c_0.

If $M(t)=t^p$, $1\le p<\infty$, then $d(w,M)=d(w,p)$ is the Lorentz sequence space. If $w_i=1$ for every $i\in\mathbb{N}$, then $d(w,M)$ is the Orlicz sequence space ℓ_M and $h_M=d_0(w,M)$.

The symbol e_i will stand for the unit vectors in $d_0(w,M)$.

We say that two basic sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in the Banach spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ respectively are C–equivalent, whenever for any real sequence $\{a_i\}_{i=1}^\infty$ we have

$$\frac{1}{C}\left\|\sum_{i=1}^\infty a_ix_i\right\|_X\le\left\|\sum_{i=1}^\infty a_iy_i\right\|_Y\le C\left\|\sum_{i=1}^\infty a_ix_i\right\|_X.$$

The basic sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ are said to be almost isometrically equivalent if for all $k\in\mathbb{N}$ the tails $\{x_i\}_{i=k}^\infty$ and $\{y_i\}_{i=k}^\infty$ are $(1+\varepsilon_k)$–equivalent, for some positive sequence$\{\varepsilon_k\}_{k=1}^\infty$, such that $\lim_{k\to\infty}\varepsilon_k=0$.

Deep results concerning the embedding of ℓ_p spaces into Orlicz–Lorentz sequence spaces are obtained in [9]. It is shown there that in any infinite dimensional subspace X of $d_0(w,M)$ there is an almost isometrically equivalent copy either of c_0 or to some Orlicz sequence space h_ψ, for some $\psi\in C_M$. Moreover it is shown in [9] that the same result as like as for Orlicz sequence spaces hold for the embedding of the ℓ_p spaces i.e. $\ell_p\hookrightarrow d_0(w,M)$ iff $p\in[\alpha_M,\beta_M]$.

If $\{v_n\}_{n=1}^\infty$ is a basis of a Banach space $(X,\|\cdot\|)$, and $|||\cdot|||$ is a norm equivalent to the given norm $\|\cdot\|$. We say that $\{v_n\}_{n=1}^\infty$ is monotone with respect to $|||\cdot|||$ if

$$\left|\left|\left|\sum_{n=1}^k a_nv_n\right|\right|\right|\le\left|\left|\left|\sum_{n=1}^{k+1} a_nv_n\right|\right|\right|$$

for every real sequence $a=\{a_n\}_{n=1}^\infty$ and for all $k\in\mathbb{N}$.

Theorem 1. *([11]) Let $\{v_n\}_{n=1}^{\infty}$ be a shrinking basis of a Banach space $(X, \|\cdot\|)$. The following are equivalent:*
a) X is isomorphically polyhedral.
b) There exists an equivalent norm $|||\cdot|||$ on X such that $\{v_n\}_{n=1}^{\infty}$ is a monotone basis with respect to $|||\cdot|||$, and for all $\sum_{n=1}^{\infty} a_n v_n \in X$, there exists $m \in \mathbb{N}$ such that

$$\left|\left|\left|\sum_{n=1}^{\infty} a_n v_n\right|\right|\right| = \left|\left|\left|\sum_{n=1}^{m} a_n v_n\right|\right|\right|.$$

By the Remark following Theorem 1 in [11] it follows that Theorem 1 holds also if the shrinking basis is replaced by an unconditional one.

According to a result from [1] we have the following

Theorem 2. *([1]) Every separable isomorphically polyhedral Banach space X admits an equivalent analytic norm.*

Definition 2.1. *Let U be an open, convex and bounded subset of a Banach space X, f be a real function on U. We say that f is weakly sequentially continuous (wsc) if it maps weakly Cauchy sequences from U into convergent ones.*

The next theorem, obtained in [8] gives a sufficient condition so that Banach spaces with an equivalent analytic norm to be isomorphically polyhedral.

Theorem 3. *([8]) Let $(X, \|\cdot\|)$ be a Banach space, where $\|\cdot\|$ is an analytic norm. If all polynomials on X are wsc, then X is separable and isomorphically polyhedral.*

Let us recall that a Banach space X with an unconditional basis is said to satisfy an upper p–estimate, $p > 1$, if for some constant $C > 0$ holds

$$\left\|\sum_{i=1}^{n} u_i\right\| \leq C \left(\sum_{i=1}^{n} \|u_i\|^p\right)^{1/p},$$

whenever u_i are disjointly supported in X.

We finish the preliminaries with the following

Lemma 2.1. *([7], [8]) Let X be a Banach space with an unconditional basis satisfying an upper p–estimate. Then all polynomials of degree $n < p$ on X are wsc.*

3. Main Result

Theorem 4. *Let M be an Orlicz function without Δ_2–condition at zero. Then the Orlicz–Lorentz sequence spaces $d_0(w,M)$ admits an equivalent analytic norm iff it is isomorphically polyhedral.*

4. Auxiliary Results

The next Proposition shows that results as like as the results obtained in [11] hold for characterizing the c_0–saturated Orlicz–Lorentz sequence spaces.

Proposition 4.1. *Let M be a non-degenerate Orlicz function, then the following are equivalent:*
a) $d_0(w,M)$ is c_0–saturated;
b) $d_0(w,M)$ does not contain an isomorphic copy of ℓ_p for any $p \in [1,\infty)$;
c) for all $q \in [1,+\infty)$ holds

$$\sup_{0<u,v\leq 1} \frac{M(uv)}{M(u)v^q} < \infty.$$

Proof. Clearly a) implies b).

Let now b) holds, but a) fails, then there exists an infinite dimensional closed subspace Y of $d_0(w,M)$, which contains no isomorphic copy of c_0. According to [9] Y has a subspace Z, which is almost isometrically equivalent either to c_0 or to h_ψ for some $\psi \in C_M$. As Y does not contain an isomorphic copy of c_0, the Z is almost isometrically equivalent to h_ψ. By the assumption that Y has no an isomorphic copy of c_0 and by [12](Theorem 4.a.9) it follows that h_ψ contains an isomorphic copy of ℓ_p for some $p \in [1,\infty)$, which is a contradiction.

Let now b) holds i.e. there is no isomorphic copy of ℓ_p in $d_0(w,M)$ for any $p \in [1,\infty)$. According to [9] $\ell_p \hookrightarrow d_0(w,M)$ iff $p \in [\alpha_M, \beta_M]$. Therefore $\alpha_M = \infty$ and thus for any $q \in [1,\infty)$ the inequality $\sup_{0<u,v\leq 1} \frac{M(uv)}{M(u)v^q} < \infty$ holds.

Let c) holds. Then $\alpha_M = \beta_M = \infty$. Let Y be an arbitrary infinite dimensional subspace of $d_0(w,M)$. According to [9] Y has a subspace Z, which is almost isometrically equivalent either to c_0 or to h_ψ for some function $\psi \in C_M$. If Z is almost isometrically equivalent to c_0, then there is an isomorphic copy of c_0 in Y. Let suppose that Z is not almost isometrically equivalent to c_0, then Z is almost isometrically equivalent to h_ψ for some function $\psi \in C_M$.

It is easy to see that $\alpha_\psi = \infty$, because $C_M \subseteq C_{M,1}$ and for every $\varphi \in C_{M,1}$ it is easy to show, that $[\alpha_\varphi, \beta_\varphi] \subseteq [\alpha_M, \beta_M]$. Consequently $c_0 \hookrightarrow h_\psi \hookrightarrow Y$. $\square$

The next Lemma will be needed in the sequel in order to apply Theorem 3.

Lemma 4.1. *Let M be an Orlicz function. If* $\sup_{0<u,v\leq 1} \dfrac{M(uv)}{u^p M(v)} < \infty$, *for some $p > 1$, then the Orlicz–Lorentz sequence spaces $d_0(w, M)$ has an upper p–estimate.*

Proof. Let $\sup_{0<u,v\leq 1} \dfrac{M(uv)}{u^p M(v)} \leq C_1 < \infty$. WLOG we may assume that $C_1 > 1$. Let $\{A_k\}_{k=1}^n$, $A_k \subset \mathbb{N}$, $A_j \cap A_k = \emptyset$, for every $j \neq k$. Denote by $a_k = |A_k|$ and $S = \sum_{k=1}^n a_k$. If $a_k = +\infty$ for some $k = 1, \dots, n$, then $S = +\infty$. Let $u_k = \sum_{i\in A_k} x_i e_i$, $k = 1, \dots, n$. Let $\{x^*_{ki}\}_{i=1}^{a_k}$ be a decreasing rearrangement of the sequence $\{|x_i|\}_{i\in A_k}$, for every $k = 1, 2, \dots, n$ and $\{x^*_i\}_{i=1}^S$ be a decreasing rearrangement of the sequence $\{\{|x_i|\}_{i\in A_k}\}_{k=1}^n$.

By the definition of the Luxemburg norm in $d_0(w, M)$ we have

$$\left\|\sum_{i=1}^n u_i\right\| = \inf\left\{\lambda > 0 : \sum_{i=1}^S w_i M\left(\frac{x^*_i}{\lambda}\right) \leq 1\right\} \quad \text{and}$$

$$\|u_k\| = \inf\left\{\lambda > 0 : \sum_{i=1}^{a_k} w_i M\left(\frac{x^*_{ki}}{\lambda}\right) \leq 1\right\}$$

for every $k = 1, 2, \dots, n$. Therefore by the chain of the inequalities

$$\sum_{i=1}^S w_i M\left(\frac{x^*_i}{C_1^{1/p}\left(\sum_{j=1}^n \|u_j\|^p\right)^{\frac{1}{p}}}\right) \leq \sum_{k=1}^n \sum_{i=1}^{a_k} w_i M\left(\frac{x^*_{ki}\|u_k\|}{C_1^{1/p}\left(\sum_{j=1}^n \|u_j\|^p\right)^{\frac{1}{p}} \|u_k\|}\right)$$

$$\leq \sum_{k=1}^n \sum_{i=1}^{a_k} w_i \frac{\|u_k\|^p}{\sum\limits_{i=1}^n \|u_i\|^p} M\left(\frac{x^*_{ki}}{\|u_k\|}\right) = \sum_{k=1}^n \frac{\|u_k\|^p}{\sum\limits_{i=1}^n \|u_i\|^p} \sum_{i=1}^{a_k} w_i M\left(\frac{x^*_{ki}}{\|u_k\|}\right) \leq 1$$

we get that $\|\sum_{i=1}^n u_i\| \leq C\left(\sum_{i=1}^n \|u_i\|^p\right)^{1/p}$, where $C = C_1^{1/p}$. $\square$

Proposition 4.2. *The Orlicz–Lorentz sequence space $d_0(w, M)$ is separable.*

Proof. Indeed according to Proposition 2.1 the unit vectors $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis in $d_0(w, M)$ and therefore the set $M = \left\{\sum_{i=1}^{n} p_i e_i : p_i \in \mathbb{Q}, n \in \mathbb{N}\right\}$ is a countable dense set in $d_0(w, M)$. □

5. Proof of Main Result

Proof. If $d_0(w, M)$ is isomorphically polyhedral, then according to Theorem 2 and Proposition 4.2 it admits an equivalent analytic norm.

Let now $M \notin \Delta_2(0)$ and let there exists an equivalent analytic norm in $d_0(w, M)$. According to [9] $\ell_p \hookrightarrow d_0(w, M)$ iff $p \in [\alpha_M, \beta_M]$, therefore $\alpha_M = \beta_M = +\infty$. By Lemma 4.1 $d_0(w, M)$ has an upper p–estimate for every $p > 1$ and finally according to Theorem 3 we get that $d_0(w, M)$ is isomorphically polyhedral. □

Remark: A natural question arises to characterize the Orlicz–Lorentz sequence spaces $d_0(w, M)$, which admit an equivalent analytic norm as like as it was done in [8]. Let us mention that if $M \in \Delta_2(0)$ and there exists an equivalent analytic norm in $d_0(w, M)$, then according to [9] we get that $\alpha_M = \beta_M \in \{2n\}_{n\in\mathbb{N}} \cup \{+\infty\}$.

6. A class of Orlicz–Lorentz sequence spaces $d_0(w, M)$, that admit an equivalent analytic norm

Proposition 6.1. *Let M be a non–degenerate Orlicz function, such that there exists a finite number $\lambda > 1$ satisfying*

$$\lim_{t\to 0} \frac{M(\lambda t)}{M(t)} = \infty. \tag{2}$$

Then $d_0(w, M)$ is isomorphically polyhedral.

Proof. For all $k \in \mathbb{N}$ let define:

$$b_k = \inf\left\{\frac{M(\lambda t)}{M(t)} : 0 < t \le M^{-1}\left(\frac{1}{W(k)}\right)\right\}.$$

By (2) and the fact that $\lim_{k\to\infty} W(k) = \infty$ it follows that $\{b_k\}_{k=1}^{\infty}$ is an increasing sequence and $\lim_{k\to\infty} b_k = \infty$. Thus there exists a sequence $\{\eta_k\}_{k=1}^{\infty}$ such that $\eta_k \searrow 1$ and $\eta_k > \frac{1}{1-\frac{1}{b_{k+1}}}$.

For $x = \{x_n\}_{n=1}^{\infty} \in d_0(w, M)$ define a norm $|||\cdot|||$ in $d_0(w, M)$ by

$$|||\{x_n\}||| = \sup_k \eta_k \|(x_1^*, x_2^*, \dots, x_k^*, 0, 0, \dots)\|, \tag{3}$$

where $\|\cdot\|$ is the Luxemburg's norm in $d_0(w, M)$.

Claim 6.1. *The norm $|||\cdot|||$ is an equivalent to the Luxemburg norm in $d_0(w, M)$.*

Proof of Claim 6.1: The proof follows by the inequalities:

$$|||\{x_n\}||| = \sup_k \eta_k \|(x_1^*, \dots, x_k^*, 0, 0, \dots)\| \leq \eta_1 \|\{x_n\}\|$$

and

$$|||\{x_n\}||| = \sup_k \eta_k \|(x_1^*, \dots, x_k^*, 0, 0, \dots)\| \geq \sup_k \|(x_1^*, \dots, x_k^*, 0, 0, \dots)\| \geq \|\{x_n\}\|.$$

□

Claim 6.2. *The unit vector basis $\{e_i\}_{i=1}^{\infty}$ is a monotone basis in $d_0(w, M)$ with respect to $|||\cdot|||$.*

Proof of Claim 6.2: Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary real sequence and $k \in \mathbb{N}$ be an arbitrary chosen. We need to show that

$$\left|\left|\left|\sum_{i=1}^{k} x_i e_i\right|\right|\right| \leq \left|\left|\left|\sum_{i=1}^{k+1} x_i e_i\right|\right|\right|.$$

Let denote by $\{a_i\}_{i=1}^{k}$ the decreasing rearrangement of the set $\{|x_i|\}_{i=1}^{k}$ and by $\{b_i\}_{i=1}^{k+1}$ the decreasing rearrangement of the set $\{|x_i|\}_{i=1}^{k+1}$. It is easy to see that $b_i \geq a_i$ for every $i = 1, \dots, k$. Thus

$$\begin{aligned} \left|\left|\left|\sum_{i=1}^{k} x_i e_i\right|\right|\right| &= \sup_{i=1,\dots,k} \eta_i \|(a_1, \dots, a_i, 0 \dots)\| \leq \sup_{i=1,\dots,k} \eta_i \|(b_1, \dots, b_i, 0 \dots)\| \\ &\leq \sup_{i=1,\dots,k+1} \eta_i \|(b_1, \dots, b_i, 0 \dots)\| = \left|\left|\left|\sum_{i=1}^{k+1} x_i e_i\right|\right|\right|. \end{aligned}$$

□

To finish the proof of the theorem it suffices to show that $|||\cdot|||$ satisfies the condition b) in Theorem 1.

Claim 6.3. *For any positive decreasing sequence* $x = \{x_i\}_{i=1}^{\infty} \in d_0(w, M)$ *there is* $k \in \mathbb{N}$ *such that*

$$\|x\| \leq \eta_k \|(x_1, x_2, \dots, x_k, 0, 0, \dots)\|. \tag{4}$$

Proof of Claim 6.3: Assume otherwise i.e. for any $k \in \mathbb{N}$ the inequality (4) does not hold. WLOG we may assume that $\|x\| = 1$. Then

$$\sum_{i=1}^{\infty} w_i M(x_i) = 1 \tag{5}$$

and

$$\sum_{i=1}^{k} w_i M(\eta_k x_i) \leq 1 \tag{6}$$

holds for every $k \in \mathbb{N}$. Notice that by the fact that $\{x_i\}_{i=1}^{\infty}$ is a decreasing sequence and $\eta_k \geq 1$ it follows from inequality (6) that $x_k \in \left[0, M^{-1}\left(1/W(k)\right)\right]$.

By the fact that $\{e_i\}_{i=1}^{\infty}$ is a basis in $d_0(w, M)$ it follows that there exists $m \in \mathbb{N}$ such that $\|(0, \dots, 0, x_m, x_{m+1}, \dots)\| \leq 1/\lambda$ i.e. $\sum_{i=1}^{\infty} w_i M(\lambda x_{m+i-1}) \leq 1$.

For any $i \in \mathbb{N}$, $i \geq m$ the inequality $\dfrac{M(\lambda x_i)}{M(x_i)} \geq b_m$ holds. Now we can write the chain of inequalities:

$$\begin{aligned} 1 &= \sum_{i=1}^{\infty} w_i M(x_i) = \sum_{i=1}^{m-1} w_i M(x_i) + \sum_{i=m}^{\infty} w_i M(x_i) \\ &\leq \frac{1}{\eta_{m-1}} \sum_{i=1}^{m-1} w_i M(\eta_{m-1} x_i) + \frac{1}{b_m} \sum_{i=m}^{\infty} w_i M(\lambda x_i) \\ &\leq \frac{1}{\eta_{m-1}} + \frac{1}{b_m} \sum_{i=1}^{\infty} w_i M(\lambda x_{m+i-1}) \\ &\leq \frac{1}{\eta_{m-1}} + \frac{1}{b_m} < 1 \end{aligned}$$

which is a contradiction. Hence (4) holds for some $k \in \mathbb{N}$. □

Now for a general element $x = \{x_i\}_{i=1}^{\infty} \in d_0(w, M)$, choose $m \in \mathbb{N}$ such that

$$\|\{x_i\}\| = \|\{x_i^*\}\| \leq \eta_m \|(x_1^*, \dots, x_m^*, 0, 0, \dots)\|.$$

Since $\lim_{k\to\infty} \eta_k \|(x_1^*, \dots, x_k^*, 0, 0, \dots)\| = \|\{x_i\}\|$ by Claim 6.3 the supremum in (3) is attained at some $j \in \mathbb{N}$. Choose $i \in \mathbb{N}$ so that $\{x_1^*, \dots, x_j^*\} \subset \{|x_1|, \dots, |x_i|\}$, then

$$|||(x_1, \dots, x_i, 0, 0, \dots)||| \geq \eta_j \|(x_1^*, \dots, x_j^*, 0, 0, \dots)\| = |||\{x_n\}|||.$$

Since the reverse inequality $|||\{x_n\}||| \geq |||(x_1, \dots, x_i, 0, 0, \dots)|||$ is obvious it follows that

$$|||\{x_n\}||| = |||(x_1, \dots, x_i, 0, 0, \dots)|||.$$

□

Corollary 6.1. *Let M be a non–degenerate Orlicz function, such that there exists a finite number $\lambda > 1$ satisfying*

$$\lim_{t\to 0} \frac{M(\lambda t)}{M(t)} = \infty.$$

Then $d_0(w, M)$ admits an equivalent analytic norm, it is c_0–saturated and it has a separable dual.

Proposition 6.2. *Let M be a non-degenerate Orlicz function. Suppose there exists a sequence $\{t_n\}_{n=1}^{\infty}$ decreasing to 0 such that*

$$\sup_{n\in\mathbb{N}} \frac{M(Kt_n)}{M(t_n)} < \infty$$

for all $K < \infty$. Then $d_0(w, M)$ is not isomorphically polyhedral.

Proof. Suppose the contrary i.e. $d_0(w, M)$ is isomorphically polyhedral. According to Theorem 1 one can obtain an equivalent norm $|||\cdot|||$ on $d_0(w, M)$ as prescribed by part b). Fix $\alpha > 0$ so that $|||x||| \leq \alpha$ implies $\|x\| \leq 1$. Choose an arbitrary sequence $\eta_k \searrow 1$. Let

$$n_1 = \min\{n \in \mathbb{N} : \eta_1 |||t_n e_1||| \leq \alpha\}.$$

If $n_1 \leq n_2 \leq \dots \leq n_k$ are chosen so that $\eta_i \left|\left|\left|\sum_{j=1}^{i} t_{n_j} e_j\right|\right|\right| \leq \alpha$, for every $i = 1, 2, \dots, k$, then by $\eta_{k+1} < \eta_k$ it follows that $\eta_{k+1} \left|\left|\left|\sum_{j=1}^{k} t_{n_j} e_j\right|\right|\right| < \alpha$.

Hence $\left\{n \geq n_k : \eta_{k+1} \left|\left|\left|\sum_{j=1}^{k} t_{n_j} e_j + t_n e_{k+1}\right|\right|\right| < \alpha\right\} \neq \emptyset$.

Now define

$$n_{k+1} = \min\left\{n \geq n_k : \eta_{k+1} \left|\left|\left|\sum_{j=1}^{k} t_{n_j} e_j + t_n e_{k+1}\right|\right|\right| \leq \alpha\right\}. \tag{7}$$

This inductively defines a nondecreasing sequence of naturals $\{n_k\}_{k=1}^{\infty}$, such that

$$\eta_k \left|\left|\left| \sum_{j=1}^{k} t_{n_j} e_j \right|\right|\right| \leq \alpha \tag{8}$$

for every $k \in \mathbb{N}$, fulfilling the minimality condition (7). In particular $\left|\left|\left|\sum_{j=1}^{k} t_{n_j} e_j\right|\right|\right| \leq \alpha$. Therefore $\left\|\sum_{j=1}^{k} t_{n_j} e_j\right\| \leq 1$, i.e. $\sum_{j=1}^{k} w_j M(t_{n_j}) \leq 1$ holds for every $k \in \mathbb{N}$.

For any $\lambda < \infty$ and for any $k \in \mathbb{N}$ the chain of inequalities

$$\sum_{j=1}^{k} w_j M(\lambda t_{n_j}) = \sum_{j=1}^{k} w_j \frac{M(\lambda t_{n_j})}{M(t_{n_j})} M(t_{n_j}) \leq$$

$$\leq \sup_{m\in\mathbb{N}} \frac{M(\lambda t_m)}{M(t_m)} \sum_{j=1}^{k} w_j M(t_{n_j}) \leq \sup_{m\in\mathbb{N}} \frac{M(\lambda t_m)}{M(t_m)} < \infty$$

holds. Hence $x = \sum_{j=1}^{\infty} t_{n_j} e_j \in d_0(w, M)$. Clearly

$$|||x||| = \lim_{k\to\infty} \left|\left|\left| \sum_{j=1}^{k} t_{n_j} e_j \right|\right|\right| \leq \alpha.$$

Claim 6.4. $|||x||| = \alpha$.

Proof of Claim 6.4: Suppose otherwise i.e. $|||x||| = \beta < \alpha$ for some β. Since $\{e_n\}_{n=1}^{\infty}$ is a monotone basis with respect to $|||\cdot|||$ we get $\left|\left|\left|\sum_{j=1}^{k} t_{n_j} e_j\right|\right|\right| \leq \beta < \alpha$ for every $k \in \mathbb{N}$.

By $x = \sum_{j=1}^{\infty} t_{n_j} e_j \in d_0(w, M)$ it follows that $t_{n_j} \searrow 0$. We can choose $i_0 \in \mathbb{N}$ such that $|||t_{n_{i_0}} e_j||| \leq \alpha - \beta$ holds for every $j \in \mathbb{N}$. Then by the minimality condition (7) and the inequalities:

$$\left|\left|\left| \sum_{j=1}^{i_0} t_{n_j} e_j + t_{n_{i_0}} e_{i_0+1} \right|\right|\right| \leq \left|\left|\left| \sum_{j=1}^{i_0} t_{n_j} e_j \right|\right|\right| + |||t_{n_{i_0}} e_{i_0+1}||| \leq \beta + \alpha - \beta = \alpha$$

it follows that $n_{i_0+1} = n_{i_0}$. Similarly $n_j = n_{i_0}$ for all $j \geq i_0$ which contradicts with the convergence of $x = \sum_{j=1}^{\infty} t_{n_j} e_j \in d_0(w, M)$. □

Now by Claim 6.4 and (8) the inequalities

$$|||x||| = \alpha \geq \eta_k \left|\left|\left|\sum_{j=1}^{k} t_{n_j} e_j\right|\right|\right| > \left|\left|\left|\sum_{j=1}^{k} t_{n_j} e_j\right|\right|\right|$$

hold for every $k \in \mathbb{N}$, which is a contradiction with the choice of the norm $|||\cdot|||$. □

Remark: Using the construction of Orlicz functions from [11] (Proposition 7, Theorem 8) we get that there exists an Orlicz–Lorentz sequence space $d_0(w, M)$, with generating Orlicz function $M \notin \Delta_2(0)$, which is not isomorphically polyhedral and therefore without any equivalent analytic norm.

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ВЪРХУ ЕКВИВАЛЕНТНИ АНАЛИТИЧНИ НОРМИ В РЕДИЧНИ ПРОСТРАНСТВА НА ОРЛИЧ–ЛОРЕНЦ

Б. Златанов

Резюме. Доказли сме, че ако пораждащата функция на Орлич M не удовлетворява Δ_2–условието в нулата, тогава съществуването на еквивалентна аналитична норма в редичното пространство на Орлич–Лоренц $d_0(w, M)$ е еквивалентно на това $d_0(w, M)$ да е изоморфно полиедрално. Показали сме, че ако $\lim_{t\to 0} \frac{M(\lambda t)}{M(t)} = \infty$ за някое $\lambda > 1$, тогава редичното пространство на Орлич–Лоренц $d_0(w, M)$ е изоморфно полиедрално банахово пространство и следователно в него съществува еквивалентна аналитична норма, то е наситено с c_0 и спрегнатото му пространство е сепарабелно. Направили сме пълна характеризация на редичните пространства на Орлич–Лоренц, които са наситени с c_0.