ТРАДИЦИИ, ПОСОКИ, ПРЕДИЗВИКАТЕЛСТВА
Юбилейна национална научна конференция с международно участие
Смолян, 19–21 октомври, 2012

ON THE CONTINUOUS DEPENDENCE OF THE SWITCHING MOMENTS OF TRAJECTORIES

Rumyana Chukleva[1], Angel Dishliev[2]

[1] Technical University Sofia, branch Plovdiv, Bulgaria,
[2] University of Chemical Technology and Metallurgy, Sofia, Bulgaria
[2] dishliev@uctm.edu

ВЪРХУ НЕПРЕКЪСНАТА ЗАВИСИМОСТ НА ПРЕВКЛЮЧВАЩИ МОМЕНТИ НА ТРАЕКТОРИИ

Румяна Чуклева[1], Ангел Дишлиев[2]

[1] Технически университет, Филиал Пловдив, България,
[2] Химикотехнологичен и металургичен университет, София, България
[2] dishliev@uctm.edu

Резюме. *Обект на изследване е начална задача за нелинейни неавтономни системи диференциални уравнения. Във фазовото пространство на системата е зададено така нареченото превключващо множество* Φ. *Намерени са достатъчни условия за:*

1. *Среща на траекторията на задачата с множеството* Φ*;*
2. *Оценка на времето за достигане на превключващото множество;*
3. *Непрекъсната зависимост на първия момент на среща на траекторията с превключващото множество относно началното условие.*

Получените резултати се прилагат при изследване на качествата на решенията на импулсни диференциални уравнения. С такива уравнения се моделират динамични процеси, които са подложени на „кратковременни” външни въздействия.

Ключови думи: *неавтономни нелинейни диференциални уравнения, превключващо множество, импулсни ефекти*

The behavior of the solution of initial problem of nonlinear non-autonomous systems of ordinary differential equations is investigated in this paper:

$$\frac{dx}{dt} = f(t,x), \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

where the function $f: R^+ \times D \to R^n$; the phase space D is non empty domain in R^n; the initial point $(t_0, x_0) \in R^+ \times D$. The solution of problem above is denoted by $x(t; t_0, x_0)$, and the corresponding trajectory by $\gamma(t_0, x_0) = \{x(t; t_0, x_0),\ t \ge t_0\}$.

We introduce the function $\varphi: D \to R$, which in the general case is nonlinear and the corresponding non empty set $\Phi = \{x \in D, \varphi(x) = 0\}$. In the current study, we determine the conditions, under which the trajectory $\gamma(t_0, x_0)$ meets the set Φ, i.e. it cancels function φ. Furthermore, the function φ and the set Φ are named switching function and switching set, respectively. For the initial point, we assume that $\varphi(x_0) \neq 0$.

Let t_1 be the first moment after t_0, in which the solution $x(t; t_0, x_0)$ meets the switching set Φ, i.e. there is

$$\varphi(x(t;t_0,x_0)) \neq 0 \quad for\ t_0 \leq t < t_1 \quad and \quad \varphi(x(t_1;t_0,x_0)) = 0.$$

The moment t_1 is a switching moment.

Consider the initial condition

$$x(t_0^*) = x_0^* \tag{3}$$

where $(t_0{}^*, x_0{}^*) \in R^+ \times D$ and $\varphi(x_0^*) \neq 0$. If $(t_0^*, x_0^*) \neq (t_0, x_0)$ is satisfied, then in the common case, the corresponding switching moment t_1^* of trajectory $\gamma(t_0^*, x_0^*)$ (if exists) is different from t_1. In other words, the effects on the initial condition of trajectory reflect on the switching moment.

The following notations are introduced:

- $\|.\|$ and $\langle .,. \rangle$ are the Euclidean norm and the scalar product in R^n, respectively. For the points $x = (x^1, x^2, ..., x^n)$, $y = (y^1, y^2, ..., y^n) \in R^n$, we have

$$\langle x, y \rangle = \left(x^1 y^1 + x^2 y^2 + ... + x^n y^n\right)^{1/2},$$

$$\|x\| = \left(\langle x, x \rangle\right)^{1/2} = \left(\left(x^1\right)^2 + \left(x^2\right)^2 + ... + \left(x^n\right)^2\right)^{1/2};$$

- The distance between non empty sets $A, B \subset R^n$ is defined by the equality

$$\rho(A, B) = \inf\left\{\|x_A - x_B\|;\ x_A \in A,\ x_B \in B\right\};$$

- In particular, the distance between the point $x_0 \in R^n$ and the set $A \subset R^n$ satisfies the equality

$$\rho(x_0, A) = \inf\left\{\|x_0 - x_A\|;\ x_A \in A\right\}.$$

Definition 1. *We say that, the switching moment t_1 of the trajectory of initial problem (1) depends continuously on the initial condition if*

$$(\forall \omega > 0)(\exists \delta = \delta(\omega) > 0) : (\forall t_0^* \in R^n, |t_0^* - t_0| < \delta)(\forall x_0^* \in D, \|x_0^* - x_0\| < \delta)$$
$$\Rightarrow |t_1^* - t_1| < \omega.$$

The next conditions are introduced:

H1. *The function* $f \in C[R^+ \times D, R^n]$.

H2. *A constant* $C_{Lip\varphi} > 0$ *exists such that*

$$(\forall x', x'' \in D) \Rightarrow \|\varphi(x') - \varphi(x'')\| \leq C_{Lip\varphi}\|x' - x''\|.$$

H3. *The function* $\varphi \in C^1[D, R]$.

H4. *A constant* $C_{grad\varphi} > 0$ *exists such that*

$$(\forall x \in D) \Rightarrow \|grad\varphi(x)\| \leq C_{grad\varphi}.$$

H5. *The following inequalities are valid:*

$$\varphi(x).\langle grad\varphi(x), f(t,x) \rangle < 0, (t,x) \in R^+ \times D.$$

H6. *A constant* $C_{\langle grad\varphi, f\rangle} > 0$ *exists such that*

$$\left(\ \forall (t,x) \in R^{+} \times D\ \right) \Rightarrow \left|\langle grad\varphi(x), f(t,x)\rangle\right| \geq C_{\langle grad\varphi, f\rangle}.$$

H7. *For any point* $(t_0, x_0) \in R^{+} \times D$, *the solution of initial problem (1), (2) exists and it is unique for* $t \geq t_0$.

The next theorems are valid:

Theorem 1. *Let the conditions H1, H3, H5, H6 and H7 be fulfilled. Then the trajectory of problem (1), (2), meets the switching set* Φ.

Proof. From condition H5 it follows that one of the following two cases is satisfied:
Case 1. $\varphi(x) < 0$ for $x \in D$ and $\langle grad\varphi(x), f(t,x)\rangle > 0$ for $(t,x) \in R^{+} \times D$;
Case 2. $\varphi(x) > 0$ for $x \in D$ and $\langle grad\varphi(x), f(t,x)\rangle < 0$ for $(t,x) \in R^{+} \times D$.

Let us consider Case 1. The other case is considered by analogy. We introduce the function

$$\phi(t) = \varphi\left(x(t;t_0,x_0)\right) = \varphi\left(x^1(t;t_0,x_0), x^2(t;t_0,x_0), \ldots, x^n(t;t_0,x_0)\right),$$

which is defined for $t \geq t_0$. In this case, we have

$$\phi(t_0) = \varphi\left(x(t_0;t_0,x_0)\right) = \varphi(x_0) < 0.$$

According to condition H6, it is satisfied

$$\begin{aligned}
\frac{d}{dt}\phi(t) &= \frac{\partial}{\partial x^1}\varphi\left(x(t;t_0,x_0)\right)\frac{d}{dt}x^1(t;t_0,x_0) \\
&\quad + \frac{\partial}{\partial x^2}\varphi\left(x(t;t_0,x_0)\right)\frac{d}{dt}x^2(t;t_0,x_0) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&\quad + \frac{\partial}{\partial x^n}\varphi\left(x(t;t_0,x_0)\right)\frac{d}{dt}x^n(t;t_0,x_0) \\
&= \frac{\partial}{\partial x^1}\varphi\left(x(t;t_0,x_0)\right) f^1\left(t, x(t;t_0,x_0)\right) \\
&\quad + \frac{\partial}{\partial x^2}\varphi\left(x(t;t_0,x_0)\right) f^2\left(t, x(t;t_0,x_0)\right) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&\quad + \frac{\partial}{\partial x^n}\varphi\left(x(t;t_0,x_0)\right) f^n\left(t, x(t;t_0,x_0)\right) \\
&= \left\langle grad\varphi\left(x(t;t_0,x_0)\right),\ f\left(t, x(t;t_0,x_0)\right)\right\rangle \\
&\geq C_{\langle grad\varphi, f\rangle} = const > 0.
\end{aligned}$$

Using the fact $\phi(t_0) < 0$ *and* $\frac{d}{dt}\phi(t) = const > 0$ for $t \geq t_0$, it follows that there exists a point $t_1 > t_0$ such that $\varphi\left(x(t_1;t_0,x_0)\right) = \phi(t_1) = 0$. It means that at the moment t_1 the trajectory $\gamma(t_0, x_0)$ meets the set Φ.

□

Corollary 1. *Let the conditions H1, H3, H5, H6 and H7 be satisfied. Then the trajectory* $\gamma\left(t_0^*, x_0^*\right)$ *of perturbed problem (1), (3) meets the switching set* Φ.

Theorem 2. *Let the conditions H1, H3, H5, H6 and H7 be satisfied. Then the next inequality is valid*

$$t_1 - t_0 \le \frac{1}{C_{\langle grad\varphi, f\rangle}} \left|\varphi(x_0)\right|. \tag{4}$$

Proof. According to the assumption, made at the beginning of the paragraph, it is satisfied $x_0 \in D \setminus \Phi$. (If we assume that $x_0 \in \Phi$, then $t_0 = t_1$ and $\varphi(x_0) = 0$. In this case, the inequality (4) is obvious).

Then, there exists a point $\tau,\ t_0 < \tau < t_1$ such that:

$$\begin{aligned}\left|\varphi(x_0)\right| &= \left|\varphi\left(x(t_1;t_0,x_0)\right) - \varphi\left(x(t_0;t_0,x_0)\right)\right| \\ &= \left|\left\langle grad\left(\varphi\left(x(\tau;t_0,x_0)\right)\right), f\left(\tau, x(\tau;t_0,x_0)\right)\right\rangle\right| \left|t_1 - t_0\right| \\ &\ge C_{\langle grad\varphi, f\rangle}(t_1 - t_0),\end{aligned}$$

from where, it follows (4).

□

Theorem 3. *Let the conditions H1, H2, H3, H5, H6 and H7 be satisfied. Then the next inequality is valid*

$$t_1 - t_0 \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \rho(x_0, \Phi). \tag{5}$$

Proof. Let:

- ε be an arbitrary positive constant;
- The point $x_\varepsilon \in \Phi$, i.e. $\varphi(x_\varepsilon) = 0$, be such that $\rho(x_0, x_\varepsilon) = \|x_0 - x_\varepsilon\| \le \rho(x_0, \Phi) + \varepsilon$;
- The function $\phi : [t_{0,}\infty) \to R$ and $\phi(t) = \varphi\left(x(t;t_0,x_0)\right)$; $\phi : [t_{0,}\infty) \to R$;
- Assume that $\varphi(x_0) < 0$ and $\langle grad\varphi(x), f(t,x)\rangle < 0, (t,x) \in R^+ \times D$.

As in the previous proof, it can be shown that $\phi(t_0) = \varphi(x_0) < 0$ and $\frac{d}{dt}\phi(t) \ge C_{\langle grad\varphi, f\rangle} > 0,\ t > t_0$. There is a point $\tau,$ where $t_0 < \tau < t_1$, such that

$$\left|\phi(t_1) - \phi(t_0)\right| = \left|\frac{d}{dt}\phi(\tau)\right|(t_1 - t_0) \ge C_{\langle grad\varphi, f\rangle}(t_1 - t_0).$$

From the inequality above, we obtain successively

$$\begin{aligned} t_1 - t_0 &\le \frac{1}{C_{\langle grad\varphi, f\rangle}} \left|\phi(t_1) - \phi(t_0)\right| \\ &= \frac{1}{C_{\langle grad\varphi, f\rangle}} \left|\varphi(x_0)\right| \\ &= \frac{1}{C_{\langle grad\varphi, f\rangle}} \left|\varphi(x_0) - \varphi(x_\varepsilon)\right| \\ &\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \|x_0 - x_\varepsilon\| \\ &\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \left(\rho(x_0, \Phi) + \varepsilon\right). \end{aligned}$$

As ε is an arbitrary constant, it follows that it is satisfied (5).

□

Corollary 2. *Let the conditions of the previous theorem be fulfilled. Then*

$$t_1^* - t_0^* \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \rho\left(x_0^*, \Phi\right).$$

The next theorem is a basic.

Theorem 4. *Let the conditions H1, H2, H3, H5, H6 and H7 be satisfied. Then*

$$\left(\forall \omega = const > 0\right)\left(\exists \delta = \delta\left(\omega\right) > 0\right) : \left(\forall t_0^* \in R^n, \left|t_0^* - t_0\right| < \delta\right)\left(\forall x_0^* \in D, \left\|x_0^* - x_0\right\| < \delta\right)$$

$$\Rightarrow \left|t_1^* - t_1\right| \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \omega.$$

Proof. For the convenience, we assume that the inequality $t_1 \le t_1^*$ is valid. Let ω be an arbitrary positive constant. From the theorem of continuous dependence (see Theorem 7.1, Dishliev A., Bainov D.), it follows that there is a constant $\delta > 0$, such that if the considered requirements of the theorem are valid, then

$$\rho\left(x\left(t_1; t_0^*, x_0^*\right), x\left(t_1; t_0, x_0\right)\right) = \left\|x\left(t_1; t_0^*, x_0^*\right) - x\left(t_1; t_0, x_0\right)\right\| \le \omega.$$

We have

$$\rho\left(x\left(t_1; t_0^*, x_0^*\right), \Phi\right) \le \rho\left(x\left(t_1; t_0^*, x_0^*\right), x\left(t_1; t_0, x_0\right)\right) \le \omega.$$

We apply Theorem 3 and obtain the estimate

$$\left|t_1^* - t_1\right| = t_1^* - t_1 \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \rho\left(x\left(t_1; t_0^*, x_0^*\right), \Phi\right) \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \omega.$$

□

References

Bainov D., Dishliev A., Stamova I., Lipschitz quasistability of impulsive differential – difference equations with variable inpulsive perturbations // *J. of Computational and Applied Mathematics*, Vol. 70, Issue 2, (1996), 267-277.

Benchohra M., Henderson J., Ntouyas S., Impulsive differential equations and inclusions, Contemporary Mathematics and its Applications. Hindawi Publishing Corporations, Vol. 2, (2006).

Benchohra M., Henderson J., Ntouyas S., Ouahab A., Impulsive functional differential equations with variable times and infinite delay // *International J. of Applied Mathematical Sciences*, Vol. 2, Issue 1, (2005), 130-148.

Chukleva R., Modeling using differential equations with variable structure and impulses // *International Journal of Pure and Applied Mathematics*, Vol. 72, Issue 3, (2011), 343-364.

Chukleva R., Dishliev A., Dishlieva K., Continuous dependence of the solutions of the differential equations with variable structure and impulses in respect of switching functions // *International J. of Applied Science and Technology*, Vol. 1, Issue 5, (2011), 46-59.

Chukleva R., Dishliev A., Dishlieva K., Stability of differential equations with variable structure and non fixed impulsive moments using sequences of Lyapunov’s functions // *International J. of Differential Equations and Applications*, Vol. 11, Issue 1, (2012), 57-80.

Coddington E., Levinson N., Theory of ordinary differential equations, McGraw-Hill Book Company, New York, Toronto, London, (1955).

Dishliev A., Bainov D., Continuous dependence of the solution of a system of differential equations with impulses on the impulse hypersurfaces // *J. of Math. Analysis and Applications*, Vol. 135, Issue 2, (1988), 369-382.

Dishliev A., Dishlieva K., Nenov S., Specific asymptotic properties of the solutions of impulsive differential equations. Methods and applications, Academic Publications, Ltd. (2012).

Dishlieva K., Differentiability of solutions of impulsive differential equations with respect to the impulsive perturbations // *Nonlinear Analysis Series B: Real World Applications*, Vol. 12, Issue 6, (2011), 3541-3551.

Dishlieva K., Continue dependence of the solutions of impulsive differential equations on the initial conditions and barrier curves // *Acta Mathematica Scientia*, Vol. 32, Issue 3, (2012), 1035-1052.

Hristova S., Bainov D., Applications of the monotone-iterative technique of V. Lakshmikantham for solving the initial value problem for impulsive differential-difference equations // *Rocky Mountain J. of Mathematics*, Vol. 23, Issue 2, (1993), 609-618.

Hung J., Gao W., Hung J., Variable structure control: a survey, Industrial Electronics // *IEEE Transactions* on, Vol. 40, Issue 1, (1993), 2-22.

Nenov S., Impulsive controllability and optimizations problems in population dynamics // *Nonlinear Analysis,* Vol. 36, Issue 7, (1999), 881-890.

Nieto J., Rodriges-Lopez R., Boundary value problems for a class of impulsive functional equations // *Computers & Mathematics with Applications*, Vol. 55, Issue 12, (2008), 2715-2731.

Stamova I., Stability analysis of impulsive functional differential equations, Walter de Gruyter, Berlin, New York, (2009).