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НЯКОИ ДИСКРЕТНИ НЕРАВЕНСТВА С „МАКСИМУМИ”

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SOME DISCRETE NONLINEAR INEQUALITIES WITH “MAXIMA”

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Резюме. *Решени са някои нелинейни дискретни неравенства за неизвестна функция, която е повдигната на степен. Основната характеристика на разглежданите неравенства е присъствието на максимума на неизвестната функция в предишен интервал от време, а също така и нелинейното присъствие на неизвестната функция в двете страни на неравенствата. Директното приложение на получените резултати е илюстрирано върху конкретни примери за диференчни уравнения с максимуми.*

Ключови думи: *дискретни неравенства, максимум, диференчни уравнения*

Preliminary Notes

A very useful apparatus in the theory of difference equations are difference inequalities (Agarwal 2000), (Sheng, Li 2008). These inequalities are discrete analogous of integral inequalities of Gronwall and Bihary types (Stefanova, Gluhcheva 2009), (Stefanova 2011). We will solve some discrete inequalities in which the unknown function will be involved nonlinearly with its maximum value over a previous time interval.

Let $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{N}_\alpha = \{\alpha, \alpha+1, \alpha+2, \ldots\}$, $\mathbb{N}_{[\alpha,\beta]} = \{\alpha, \alpha+1, \ldots, \beta\}$, where $\alpha < \beta$ are integers. In the proof of our main results we will use the following Lemma.

Lemma 1. (Agarwal 2000, Theorem 4.1.1) Let $u, a, b, f : \mathbb{N}_\alpha \to \mathbb{R}_+$ and for all $n \in \mathbb{N}_\alpha$ be satisfied $u(n) \le a(n) + b(n)\sum_{s=\alpha}^{n-1} f(s)u(s)$. Then for all $n \in \mathbb{N}_\alpha$ the inequality $u(n) \le a(n) + b(n)\sum_{s=\alpha}^{n-1} a(s) f(s) \prod_{r=s+1}^{n-1}\left(1 + b(r) f(r)\right)$ holds.

Main Results

Theorem 1. *Let the following conditions be fulfilled:*
1. The function $a : \mathbb{N}_0 \to \mathbb{R}_+$ *is nondecreasing.*

2. The function $b:\mathbb{N}_0 \to \mathbb{R}_+$.

3. The function $\varphi:\mathbb{N}_{[-h,-1]} \to \mathbb{R}_+$ *and* $\max\limits_{k\in\mathbb{N}_{[-h,-1]}} (\varphi(k))^p \le a(0)$, *where* $h\in\mathbb{N}_0$ *and* $p \ge 1$ *are given numbers.*

4. The functions $L, M:\mathbb{N}_0\times\mathbb{R}_+ \to \mathbb{R}_+$ *satisfy*

$$0 \le L(n,x) - L(n,y) \le M(n,y)(x-y) \quad \text{for } x \ge y,$$

$$B(n) = b(n) + M\left(n, 1-\frac{q}{p}\right) + M\left(n, 1-\frac{1}{p}\right) < 1 \quad \text{for } n\in\mathbb{N}_0, \tag{1}$$

where $q\in(0,p]$ *is a constant.*

5. The function $u:\mathbb{N}_{-h} \to \mathbb{R}_+$ *satisfies the inequalities*

$$u^p(n) \le a(n) + \sum_{s=0}^{n} \left\{ b(s)u^p(s) + L\left(s, u^q(s)\right) + L\left(s, \max_{k\in\mathbb{N}_{[s-h,s]}} u(k)\right) \right\} \text{ for } n\in\mathbb{N}_0 \tag{2}$$

$$u(n) \le \varphi(n) \qquad \text{for } n\in\mathbb{N}_{[-h,-1]}. \tag{3}$$

Then for $n\in\mathbb{N}_0$ *the inequality*

$$u(n) \le \left\{ a(n) + \frac{A(n)}{1-B(n)} + \sum_{s=0}^{n-1} A(s)B(s) \prod_{r=s}^{n} \frac{1}{1-B(r)} \right\}^{\frac{1}{p}} \tag{4}$$

holds, where

$$A(n) = \sum_{s=0}^{n} \left\{ a(s)b(s) + a(s)\left[M\left(s, 1-\frac{q}{p}\right) + M\left(s, 1-\frac{1}{p}\right) \right] + L\left(s, 1-\frac{q}{p}\right) + L\left(s, 1-\frac{1}{p}\right) \right\}. \tag{5}$$

Proof. Define a function $z:\mathbb{N}_{-h} \to \mathbb{R}_+$ by the equalities

$$z(n) = \begin{cases} \sum_{s=0}^{n} \left\{ b(s)u^p(s) + L\left(s, u^q(s)\right) + L\left(s, \max\limits_{k\in\mathbb{N}_{[s-h,s]}} u(k)\right) \right\} & \text{for } n\in\mathbb{N}_0, \\ 0 & \text{for } n\in\mathbb{N}_{[-h,-1]}. \end{cases} \tag{6}$$

From the definitions of the functions $b(n)$, $u(n)$ and $L(n,x)$ we can conclude that $z(n)$ is nondecreasing in $\mathbb{N}_{-h}$ and

$$u^p(n) \le \begin{cases} a(n) + z(n) & \text{for } n\in\mathbb{N}_0, \\ \varphi^p(n) & \text{for } n\in\mathbb{N}_{[-h,-1]}, \end{cases} \tag{7}$$

i.e. $\max\limits_{k\in\mathbb{N}_{[n-h,n]}} u^p(k) \le a(n) + z(n)$ for $n\in\mathbb{N}_0$. Using the inequality $\alpha^{\frac{1}{p}}\beta^{\frac{1}{q}} \le \frac{\alpha}{p} + \frac{\beta}{q}$, where $\alpha \ge 0$, $\beta \ge 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, we observe for $n\in\mathbb{N}_0$

$$\max_{k\in\mathbb{N}_{[n-h,n]}} u(k) \le \left\{a(n)+z(n)\right\}^{\frac{1}{p}} = \left\{a(n)+z(n)\right\}^{\frac{1}{p}} 1^{\frac{1}{p/(p-1)}} \le \frac{1}{p}\left[p - 1 + a(n) + z(n)\right] \tag{8}$$

and from (7) for $n\in\mathbb{N}_0$ we obtain

$$u^q(n) \le \left\{a(n)+z(n)\right\}^{\frac{q}{p}} = \left\{a(n)+z(n)\right\}^{\frac{1}{p/q}} 1^{\frac{1}{(p/q)/((p/q)-1)}} \le \frac{1}{p}\left[p - q + qa(n) + qz(n)\right]. \tag{9}$$

Then from (8), (9) and $q\in(0,p]$ we get

$$L\left(n,u^q(n)\right)\le L\left(n,1-\frac{q}{p}+a(n)+z(n)\right),\quad L\left(n,\max_{k\in\mathbb{N}_{[n-h,n]}}u(k)\right)\le L\left(n,1-\frac{1}{p}+a(n)+z(n)\right).\tag{10}$$

Using (6), (7), (10) and condition 4 of the theorem for $n\in\mathbb{N}_0$ we get

$$z(n)\le\sum_{s=0}^{n}\left\{b(s)\left[a(s)+z(s)\right]+\left[M\left(s,1-\frac{q}{p}\right)+M\left(s,1-\frac{1}{p}\right)\right]\left[a(s)+z(s)\right]+L\left(s,1-\frac{q}{p}\right)+L\left(s,1-\frac{1}{p}\right)\right\}$$

$$=A(n)+\sum_{s=0}^{n}B(s)z(s),$$

or

$$z(n)\le\frac{A(n)}{1-B(n)}+\frac{1}{1-B(n)}\sum_{s=0}^{n-1}B(s)z(s),$$

where $A(n)$ and $B(n)$ are defined by (5) and (1), respectively. According to Lemma 1 and using the inequality in (7) for $n\in\mathbb{N}_0$ we obtain (4).

□

Theorem 2. *Let the following conditions be fulfilled:*
1. The conditions 1, 2, 3 of Theorem 1 are satisfied.
2. The condition 4 of Theorem 1 is satisfied, where $B(n)=b(n)+M\left(n,1-\frac{1}{p}\right)<1$.
3. The function $u:\mathbb{N}_{-h}\to\mathbb{R}_+$ *satisfies the inequalities*

$$u^p(n)\le a(n)+\sum_{s=0}^{n}\left\{b(s)u^p(s)+L\left(s,\max_{k\in\mathbb{N}_{[s-h,\,s]}}u(k)\right)\right\}\quad\text{for } n\in\mathbb{N}_0,\tag{11}$$

$$u(n)\le\varphi(n)\qquad\text{for } n\in\mathbb{N}_{[-h,-1]}.\tag{12}$$

Then for $n\in\mathbb{N}_0$ *the inequality (4) holds, where*

$$A(n)=\sum_{s=0}^{n}\left\{a(s)b(s)+a(s)M\left(s,1-\frac{1}{p}\right)+L\left(s,1-\frac{1}{p}\right)\right\}.$$

In the partial case $L(n,x)=w(n)x$ we obtain:

Corollary 1. *Let the following conditions be fulfilled:*
1. The conditions 1, 2 and 3 of Theorem 1 hold.
2. The function $w:\mathbb{N}_0\to\mathbb{R}_+$ *and* $b(n)+2w(n)<1$ *for* $n\in\mathbb{N}_0$.
3. The function $u:\mathbb{N}_{-h}\to\mathbb{R}_+$ *satisfies the inequalities*

$$u^p(n)\le a(n)+\sum_{s=0}^{n}b(s)u^p(s)+w(s)u^q(s)+w(s)\max_{k\in\mathbb{N}_{[s-h,s]}}u(k)\}\quad\textit{for } n\in\mathbb{N}_0,\tag{13}$$

$$u(n)\le\varphi(n)\qquad\textit{for } n\in\mathbb{N}_{[-h,-1]}.\tag{14}$$

Then for $n\in\mathbb{N}_0$ *the inequality*

$$u(n)\le\left\{a(n)+\frac{A(n)}{1-b(n)-2w(n)}+\sum_{s=0}^{n-1}A(s)\left[b(s)+2w(s)\right]\prod_{r=s}^{n}\frac{1}{1-b(r)-2w(r)}\right\}^{\frac{1}{p}}\tag{15}$$

holds, where

$$A(n)=\sum_{s=0}^{n}\left\{a(s)b(s)+a(s)2w(s)+w(s)\left(2-\frac{1+q}{p}\right)\right\}.$$

Let additionally $p = q = 1$. In this case we obtain:

Corollary 2. *Let the conditions 1 and 2 of Theorem 1 hold.*

1. The function $\varphi : \mathbb{N}_{[-h,-1]} \to \mathbb{R}_+$ *and* $\max_{k\in\mathbb{N}_{[-h,-1]}} \varphi(k) \le a(0)$.

2. The function $w : \mathbb{N}_0 \to \mathbb{R}_+$ *and* $b(n) + w(n) < 1$ *for* $n \in \mathbb{N}_0$.

3. The function $u : \mathbb{N}_{-h} \to \mathbb{R}_+$ *satisfies the inequalities*

$$u(n) \le a(n) + \sum_{s=0}^{n} \left\{ b(s)u(s) + w(s) \max_{k\in\mathbb{N}_{[s-h,s]}} u(k) \right\} \quad \textit{for } n \in \mathbb{N}_0, \tag{16}$$

$$u(n) \le \varphi(n) \quad \textit{for } n \in \mathbb{N}_{[-h,-1]}. \tag{17}$$

Then for $n \in \mathbb{N}_0$

$$u(n) \le \frac{a(n)}{1-\left[b(n)+w(n)\right]} + \sum_{s=0}^{n-1} a(s)\left[b(s)+w(s)\right] \prod_{r=s}^{n} \frac{1}{1-\left[b(r)+w(r)\right]}. \tag{18}$$

Example 1. Let

$$\begin{aligned} &u^4(n) \le C + \sum_{s=0}^{n} \frac{1}{4^{s+1}} \left\{ u^4(s) + u^2(s) + \max_{k\in\mathbb{N}_{[s-h,s]}} u(k) \right\} \quad \text{for } n \in \mathbb{N}_0,\ C \ge 0, \\ &u(n) \le \sqrt[4]{C} \quad \text{for } n \in \mathbb{N}_{[-h,-1]}. \end{aligned} \tag{19}$$

Apply Corollary 1 for $a(n) = C$, $b(n) = w(n) = \dfrac{1}{4^{n+1}}$. In this partial case we obtain the inequalities $A(n) = \left(\dfrac{3C}{4} + \dfrac{5}{16}\right) \sum_{s=0}^{n} \dfrac{1}{4^s} \le \left(C + \dfrac{5}{12}\right)$, $B(n) = \dfrac{3}{4^{n+1}} \le \dfrac{3}{4}$ and $\dfrac{1}{1-B(n)} \le 4$. Therefore, for $n \in \mathbb{N}_0$

$$u(n) \le \sqrt[4]{C + 4\left(C + \frac{5}{12}\right) + \sum_{s=0}^{n-1} \left(C + \frac{5}{12}\right) \frac{3}{14} \prod_{r=s}^{n} 4} \le \sqrt[4]{C\left(5 + 4^{n+1}\right) + \frac{5}{3}\left(1 + 4^n\right)}.$$

Example 2. Let the inequalities

$$\begin{aligned} &u^4(n)\sqrt{u(n)} \le C + \sum_{s=0}^{n} \frac{1}{4^{s+1}} \left\{ u^4(s)\sqrt{u(s)} + \max_{k\in\mathbb{N}_{[s-h,\, s]}} u(k) \right\} \quad \text{for } n \in \mathbb{N}_0, \\ &u(n) \le \sqrt[9]{C^2} \quad \text{for } n \in \mathbb{N}_{[-h,\, -1]} \end{aligned} \tag{20}$$

hold, where $C \ge 0$ is a constant.

We will apply Theorem 2 for p=4.5, $q = 0$, $a(n) = C$, $b(n) = w(n) = \dfrac{1}{4^{n+1}}$. In this partial case we obtain the inequalities $A(n) = \left(2C + \dfrac{7}{9}\right) \sum_{s=0}^{n} \dfrac{1}{4^{s+1}} \le \dfrac{1}{3}\left(2C + \dfrac{7}{9}\right)$, $B(n) = \dfrac{2}{4^{n+1}} \le \dfrac{1}{2}$ and $\dfrac{1}{1-B(n)} = 1 + \dfrac{2}{4^{n+1}-2} \le 2$.

Therefore, for $n \in \mathbb{N}_0$

$$u(n) \le \sqrt[9]{\left\{ C + \frac{2}{3}\left(2C + \frac{7}{9}\right) + \sum_{s=0}^{n-1} \frac{1}{6}\left(2C + \frac{7}{9}\right) \prod_{r=s}^{n} 2 \right\}^2} \le \sqrt[9]{\left\{ \frac{C}{3}\left(7 + 2^{n+2}\right) + \frac{14}{27}\left(1 + 2^n\right) \right\}^2}.$$

Applications

Let the function $r:\mathbb{N}_0 \to \mathbb{N}_{-h}$ be given and there exists a positive integer h such that $: n-h \le r(n) \le n$ for $n \in \mathbb{N}_0$.

Consider the following difference equation with "maxima"

$$u^p(n) = u^p(n-1) + f\left(n, u(n), \max_{k\in\mathbb{N}_{[r(n),\,n]}} u(k)\right) \quad \text{for } n \in \mathbb{N}_0, \tag{21}$$

with initial condition

$$u(n) = \varphi(n) \qquad \text{for } n \in \mathbb{N}_{[-h,-1]}, \tag{22}$$

where $u:\mathbb{N}_{-h} \to \mathbb{R}$ and $p \in [1, +\infty)$ is a constant.

Theorem 3. (*Upper bound*) *Let the following conditions be fulfilled:*

1. The function $f:\mathbb{N}_0 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the condition* $\left| f(n,x,y) \right| \le w(n)\,|x|^q + w(n)\,|y|$, *where* $w:\mathbb{N}_0 \to (0, 0.25)$ *and* $q \in (0, p]$ *is a constant.*

2. The function $\varphi:\mathbb{N}_{[-h,-1]} \to \mathbb{R}$.

Then for $n \in \mathbb{N}_0$ *the inequality*

$$|u(n)| \le \left\{ C^p + \frac{C^p + 1 - \frac{1+q}{2p}}{1 - 2w(n)(n+1)} + \frac{1}{2}\left\{2C^p + 2 - \frac{1+q}{p}\right\} \sum_{s=0}^{n} (s+1) \prod_{r=s}^{n} \frac{1}{1-2w(r)} \right\}^{\frac{1}{p}} \tag{23}$$

holds, where $C = \left| \max_{k\in\mathbb{N}_{[-h,\,-1]}} \varphi(k) \right|$.

Proof. The function $u(n)$ for $n \in \mathbb{N}_0$ satisfies

$$\left(u(n)\right)^p = \left(\max_{k\in\mathbb{N}_{[-h,\,-1]}} \varphi(k) \right)^p + \sum_{s=0}^{n} f\left(s, u(s), \max_{k\in\mathbb{N}_{[r(s),\,s]}} u(k)\right).$$

Then for $n \in \mathbb{N}_0$ we obtain

$$|u(n)|^p \le C^p + \sum_{s=0}^{n} \left| f\left(s, u(s), \max_{k\in\mathbb{N}_{[r(s),\,s]}} u(k)\right) \right| \le C^p + \sum_{s=0}^{n} \left\{ w(s)\,|u(s)|^q + w(s) \max_{k\in\mathbb{N}_{[r(s),\,s]}} |u(k)| \right\}.$$

Then since $\mathbb{N}_{[r(s),\,s]} \subseteq \mathbb{N}_{[s-h,\,s]}$ we get for $n \in \mathbb{N}_0$

$$|u(n)|^p \le C^p + \sum_{s=0}^{n} \left\{ w(s)\,|u(s)|^q + w(s) \max_{k\in\mathbb{N}_{[s-h,\,s]}} |u(k)| \right\}. \tag{24}$$

We will apply Corollary 1 for $a(n) = C^p$, $b(n) = 0$, $L(n,x) \equiv w(n)x$. Then

$$A(n) = \left\{2C^p + 2 - \frac{1+q}{p}\right\} \sum_{s=0}^{n} w(s) \le \left\{2C^p + 2 - \frac{1+q}{p}\right\} \frac{n+1}{2}$$

and $\dfrac{1}{1-2w(n)} \le 2$ and from (24) we obtain (23).

□

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