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SOME REMARKS ON A CLASS OF BOUNDARY VALUE PROBLEMS THAT INCLUDES THE St. VENANT PROBLEM

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НЯКОИ БЕЛЕЖКИ ЗА КЛАС ГРАНИЧНИ ЗАДАЧИ, ВКЛЮЧВАЩИ ПРОБЛЕМА НА St.VENANT

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Abstract. *For solutions $u(x)$ of some boundary value problems defined in a bounded convex domain Ω of $\mathbb{R}^N, N \geq 2$, we show that their points of maximum are at distance from the boundary greater than $\dfrac{d}{2}$, where d is the inradius of Ω. Moreover for $N = 2$, a minimum principle for some combination of $u(x)$ and $|\nabla u|$ is established.*

Key words: *Minimum principles, second order elliptic boundary value problems*

1. Introduction

This note addresses the following class of a boundary value problems defined in a bounded strictly convex domain $\Omega \subset \mathbb{R}^N, N \geq 2$

$$\Delta u + \alpha u + 1 = 0, x \in \Omega, u = 0, x \in \partial\Omega. \tag{1.1}$$

In (1.1) Δ is the Laplace operator and α is a constant $\in 0, \lambda_1$, where λ_1 is the first eigenvalue of the fixed membrane problem defined as

$$\Delta\varphi_1 + \lambda_1\varphi_1 = 0, \varphi_1 > 0, x \in \Omega, \quad \varphi_1 = 0, x \in \partial\Omega. \tag{1.2}$$

We note that (1.1) coincides with the St.Venant problem when $\alpha = 0$.

Problem (1.1) has been investigated by several authors [(Bandle 1976), (Kohler-Jobin 1981), (Payne, Philippin, Proytcheva 2007)].

With $\alpha < \lambda_1$, (1.1) admits a unique classical solution $u(x)$. In (Bandle 1976), Bandle shows that for $\alpha < \lambda_1$, $v(x) := \alpha u + 1$ is nonnegative in Ω, so that $\Delta u \leq 0$ in Ω, implying by the maximum principle that $u > 0$ in Ω. We note that $v(x)$ satisfies

$$\Delta v + \alpha v = 0, x \in \Omega, \quad v = 1, x \in \partial\Omega. \tag{1.3}$$

Let as assume contrariwise that $v < 0$ at some point $P \in \Omega$. Then there exists a region $\tilde{\Omega} \subset \Omega$ such that

$$v < 0 \quad \text{in} \quad \tilde{\Omega}, \quad v = 0 \quad \text{on} \quad \partial\tilde{\Omega}. \tag{1.4}$$

It then follows from Green's second identity that

$$0 = \int_{\tilde{\Omega}} v\Delta\tilde{\varphi}_1 - \tilde{\varphi}_1\Delta v \; dx = \alpha - \tilde{\lambda}_1 \int_{\Omega} v\tilde{\varphi}_1 dx, \tag{1.5}$$

where $\tilde{\varphi}_1$ is the first eigenfunction and $\tilde{\lambda}_1$ the first eigenvalue of the fixed membrane problem in $\tilde{\Omega}$. (1.5) leads to the contradiction $\alpha = \tilde{\lambda}_1 > \lambda_1$.

In the second section of this note we show that for $\alpha \in \left(0, \frac{\pi^2}{4d^2}\right)$, the maxima of $u(x)$ are located at distance greater than $\frac{d}{2}$ from the boundary $\partial\Omega$, where d is the inradius of Ω, i.e. the radius of the greatest ball contained in Ω. In the two-dimensional case $N = 2$, we derive in Section 3 an upper bound for $\min_{\partial\Omega} \left|\nabla u\right|^2$ in terms of $u_{\max}$, valid for $\alpha \in \; 0, \lambda_1$.

2. Location of the maxima of $u(x)$

Since Ω is assumed bounded and strictly convex, it follows from (Finn 2008) p. 1343 that if $N = 2$, the level lines of $u(x)$ are convex, so that $u(x)$ has a unique critical point Q at which $u = u_{\max}$. However for $N \geq 3$ and $\alpha \neq 0$, the convexity of the level sets of $u(x)$ does not seem to be established. So we cannot exclude the possibility of several critical points of $u(x)$ if $N \geq 3$, $\alpha \neq 0$. In this section we establish the following result

Theorem 1. *If* $\alpha \in \left(0, \frac{\pi^2}{4d^2}\right)$, *where* d *is the inradius of* Ω, *then the maxima of* $u(x)$ *are at distance is greater than* $\frac{d}{2}$ *from the boundary* $\partial\Omega$.

For the proof of Theorem 1, we make use of the following upper bound for $u(x)$ established in (Payne, Philippin, Proytcheva 2007).

$$u(x) \leq \frac{1}{\alpha}\left\{\frac{\cos\left[\sqrt{\alpha} \;\; d \;\; x_0 \;\; - d \;\; x \;\;\right]}{\cos \;\; \sqrt{\alpha}\, d \;\; x_0} - 1\right\}, x \in \Omega. \tag{2.1}$$

In (2.1), $d(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$, and x_0 is any point where $u(x)$ takes its maximum value. From the inequality

$$\left(\frac{\pi^2}{4d^2}\right) < \lambda_1 \tag{2.2}$$

established for convex Ω by Hersch in (Hersch 1960) for $N = 2$, and by Sperb (Sperb 1981) for $N \geq 2$, it follows that $u(x) > 0$ in Ω as already mentioned.

Inequality (2.1) then implies

$$\cos\left[\sqrt{\alpha}\left(d(x_0) - d(x)\right)\right] > \cos\left(\sqrt{\alpha}\, d(x_0)\right), \quad x \in \Omega \tag{2.3}$$

$$\text{i.e.} \qquad \left|d(x_0) - d(x)\right| < d(x_0), \;\; x \in \Omega \tag{2.4}$$

$$\text{i.e.} \qquad d(x_0) > \frac{1}{2} d(x), \;\; x \in \Omega. \tag{2.5}$$

Since (2.5) holds for all $x \in \Omega$, we obtain the desired inequality

$$d(x_0) > \frac{1}{2} \max_{x \in \Omega} d(x) = \frac{1}{2} d. \tag{2.6}$$

3. An upper bound for $\min_{\partial\Omega} |\nabla u|^2$

In (Payne, Philippin, Proytcheva 2007) the authors showed that for Ω bounded convex in $\mathbb{R}^N, N \geq 2$ the auxiliary function $\chi(x)$, defined as

$$\chi(x) := |\nabla u|^2 + \alpha u^2 + 2u, \;\; \alpha = const \in (0, \lambda_1) \tag{3.1}$$

takes its maximum at a critical point of $u(x)$. In this section we want to show that in the particular case $N = 2$, $\chi(x)$ takes its minimum value at some point on the boundary $\partial\Omega$. This leads to the following result:

Theorem 2. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^2$. Then we have*

$$\min_{\partial\Omega} |\nabla u|^2 \leq \chi(x) := |\nabla u|^2 + \alpha u^2(x) + 2u(x), \; x \in \Omega. \tag{3.2}$$

In particular

$$\min_{\partial\Omega} |\nabla u|^2 \leq \alpha u_{\max}^2 + 2u_{\max}. \tag{3.3}$$

For the proof of Theorem 2, we show under the assumptions of Theorem 2, that $\chi(x)$ satisfies an appropriate differential equation. For convenience we write $u_{,k} := \dfrac{\partial u}{\partial x_k}$ and adopt the summation convention on repeated indices. With these conventions we have for instance

$$|\nabla u|^2 = \sum_{k=1}^{2}\left(\frac{\partial u}{\partial x_k}\right)^2 = u,_k\, u,_k\,, \quad \Delta u = \sum_{k=1}^{2}\frac{\partial^2 u}{\partial x_k^2} = u,_{kk}$$

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial^2 u}{\partial x_i \partial x_k}\right) = u,_{ik}\, u,_{ik}.$$

Differentiating (3.1) and making use of (1.1), we obtain

$$\chi,_k = 2u,_{ik}\, u,_i + 2u,_k\,(\alpha u + 1) = 2u,_{ik}\, u,_i - 2u,_k\, \Delta u \tag{3.4}$$

$$\begin{aligned}\Delta\chi &= 2u,_{ik}\, u,_{ik} + 2u,_k\,(\Delta u),_k - 2u,_k\,(\Delta u),_k - 2(\Delta u)^2 \\ &= 2u,_{ik}\, u,_{ik} - 2(\Delta u)^2.\end{aligned} \tag{3.5}$$

Making use of the following identity

$$\frac{1}{2}\left|\nabla u\right|^2 \left(u,_{ik}\, u,_{ik} - (\Delta u)^2\right) = u,_{ik}\, u,_k\, u,_{ij}\, u,_j - \Delta u u,_{ik}\, u,_i\, u,_k \tag{3.6}$$

valid in $\mathbb{R}^2$ only, we obtain

$$\Delta\chi = 4\left|\nabla u\right|^{-2}\left(u,_{ik}\, u,_k\, u,_{ij}\, u,_j - \Delta u u,_{ik}\, u,_i\, u,_k\right). \tag{3.7}$$

From (3.4) rewritten as

$$u,_{ik}\, u,_i = u,_k\, \Delta u + \frac{1}{2}\chi,_k \tag{3.8}$$

we compute

$$u,_{ik}\, u,_i\, u,_{jk}\, u,_j = (\Delta u)^2\left|\nabla u\right|^2 + u,_k\, \chi,_k + \frac{1}{4}\left|\nabla\chi\right|^2 \tag{3.9}$$

$$u,_{ik}\, u,_i\, u,_k = (\Delta u)\left|\nabla u\right|^2 + \frac{1}{2}u,_k\, \chi,_k. \tag{3.10}$$

It follows from (3.7), (3.9), (3.10) that $\chi(x)$ satisfies the differential equation:

$$\Delta\chi - \left|\nabla u\right|^{-2}\nabla\chi\left(2\Delta u\nabla u + \nabla\chi\right) = 0, \quad x \in \Omega / \{Q\}, \tag{3.11}$$

where Q is the unique critical point of u. It then follows from Hopf"s first maximum principle (Hopf 1927) that $\chi(x)$ takes its maximum and minimum values either on $\partial\Omega$ or at Q.

Finally the outward normal derivative of $\chi(x)$ on $\partial\Omega$ is given by

$$\frac{\partial\chi}{\partial n} = -2K(x)\left(\frac{\partial u}{\partial n}\right)^2 \leq 0, \quad x \in \partial\Omega \tag{3.13}$$

where $K(x)$ is the curvature of $\partial\Omega$. It then follows from Hopf 's second maximum principle (Hopf 1927) that $\chi(x)$ cannot take its maximum value on $\partial\Omega$. We then conclude that $\chi(x)$ must take its maximum value at the critical point Q, and its minimum value on $\partial\Omega$. This completes the proof of Theorem 2.

We note that the inequalities in Theorem 2 are not sharp in the sense that there is no convex plane domain Ω for which we have $\chi(x) \equiv const$.

Indeed, suppose that $\chi(x) \equiv const$ in Ω. Then the identity (3.6) takes the form

$$u_{,ik}\, u_{,ik} - (\Delta u)^2 = 0, \quad x \in \Omega \tag{3.13}$$

in view of (3.9) , (3.10) with $\nabla\chi = 0$. (3.13) may be rewritten as

$$\left(\Delta u\, u_{,k} - \frac{1}{2}\left(\left|\nabla u\right|^2\right)_{,k}\right)_{,k} = 0, \quad x \in \Omega. \tag{3.14}$$

It then follows from the divergence theorem that

$$\int_{\partial\Omega} \left(\Delta u \frac{\partial u}{\partial n} - \frac{1}{2}\frac{\partial}{\partial n}\left(|\nabla u|^2\right)\right) ds = \int_{\partial\Omega} K(x)\left(\frac{\partial u}{\partial n}\right)^2 ds = 0, \tag{3.14}$$

which cannot hold since $K(x) \geq 0$ and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$ by Hopf ‘s second maximum principle.

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