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"DEATH" OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

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ЗАГИВАНЕ НА РЕШЕНИЯ НА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ С ПРОМЕНЛИВА СТРУКТУРА И ИМПУЛСИ

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Резюме. *В тази работа се изучава специален клас нелинейни неавтономни системи обикновени диференциални уравнения с променлива структура и импулси. Дясната страна на всяка една от тези системи се избира последователно от множество* f *, което се състои от безбройно много функции. Имаме* $f=\left\{f_i=f_i\left(t,x\right),\ \ i=1,2,...\right\}$. *Основни елементи на всяка система диференциални уравнения от разглеждания тип са множеството от превключващи функции:* $\varphi=\left\{\varphi_i=\varphi_i\left(x\right),\ \ i=1,2,...\right\}$ *и множеството от импулсни функции:* $I=\left\{I_i=I_i\left(x\right),\ \ i=1,2,...\right\}$. *Всяка една от превключващите функции* φ_i *и импулсните функции* I_i *е съответна на дясната страна* f_i, $i=1,2,...$. *Поредната* i *-та промяна на дясната страна на системата (смяната на* f_i *с* f_{i+1}*) и съответното импулсно въздействие върху решението* $x\left(t_i\right)\to x\left(t_i+0\right)=x\left(t_i\right)+I_i\left(x\left(t_i\right)\right)$ *се извършват в така наречения* i *-ти по ред момент на превключване* t_i, $i=1,2,...$. *Точно в този момент решението анулира превключващата функция* φ_i*, т.е.* $\varphi_i\left(x\left(t_i\right)\right)=0,\ i=1,2,...$.*Основната цел на изследванията е да се посочат причините, при които системите диференциални уравнения с променлива структура и импулси притежават решения, които не са продължими до безкрайност. Изучен е случаят, когато непродължимостта на решенията (или както е прието да се казва „загиването“ на решенията) се дължи на импулсните въздействия.*

Ключови думи: *импулсни системи, превключващи функции, „смърт на решения"*

The object of investigation in the paper is the following initial problem

$$\frac{dx}{dt}=f_i\left(t,x\right),\ \ \left\langle a_i,x\left(t\right)\right\rangle\neq\alpha_i,\ t_{i-1}<t<t_i, \tag{1}$$

$$\left\langle a_i,x\left(t_i\right)\right\rangle=\alpha_i,\ i=1,2,..., \tag{2}$$

$$x\left(t_i+0\right)=x\left(t_i\right)+I_i\left(x\left(t_i\right)\right), \tag{3}$$

$$x(t_0) = x_0, \tag{4}$$

where

- the functions $f_i : R^+ \times D \to R^n$;
- the phase space D of system considered is non empty set in R^n;
- the vectors $a_i = (a_i^1, a_i^2, ..., a_i^n) \in R^n$ and $a_i \neq 0$;
- the constants $a_i \in R$;
- the functions $I_i : D \to R^n$;
- - $(Id + I_i) : D \to D$, Id is an identity in R^n;
- the initial point $(t_0, x_0) \in R^+ \times D$, $\langle a_1, x_0 \rangle \neq \alpha_1$.

The solution of the initial problem is a piecewise continuous function with jump discontinuouty at $t_1, t_2, ...$. This solution is continuous on the left at any point in its domain. The points $t_1, t_2, ...$ are named moments of switching. The functions I_i, $i = 1, 2, ...$, are called impulsive. As it can be seen from (1) and (2), the functions $\varphi_i(x) = \langle a_i, x \rangle - \alpha_i$ are linear, and their corresponding sets:

$$\Phi_i = \{ x \in D;\ \langle a_i, x \rangle = a_i^1 x^1 + a_i^2 x^2 + ... + a_i^n x^n = \alpha_i \},\ i = 1, 2, ...$$

are parts of the hyperplanes in phase space. The functions φ_i, $i = 1, 2, ...$, and the sets Φ_i, $i = 1, 2, ...$ are called switching functions and switching sets.

The following notations are used:

- $f = \{ f_1, f_2, ... \}$, $\varphi = \{ \varphi_1, \varphi_2, ... \}$, $I = \{ I_1, I_2, ... \}$;
- $x(t; t_0, x_0)$ is a solution of problem (1), (2), (3), (4);
- $x_i(t; t_0, x_0)$ is a solution of the problem with fixed structure and without impulses

$$\frac{dx}{dt} = f_i(t, x),\ x(t_0) = x_0,\ i = 1, 2, ...; \tag{5}$$

- the curve $\gamma(t_0, x_0) = \{ x(t; t_0, x_0),\ t \in J(t_0, x_0, f) \}$ is the trajectory of the studied problem, where $J(t_0, x_0, f)$ is the maximum interval of existence of the solution;
- the curve $\gamma_i(t_0, x_0) = \{ x_i(t; t_0, x_0),\ t \in J(t_0, x_0, f_i) \}$ is the trajectory of problem (5), where $J(t_0, x_0, f_i)$ is the maximum interval of existence of the solution, $i = 1, 2, ...$;
- $\| . \|$ and $\langle .,. \rangle$ are the Euclidean norm and the scalar product in R^n, respectively.

Further, we will use the following conditions:

H1. *The functions* $f_i \in C[R^+ \times D, R^n]$, $i = 1, 2, ...$.

H2. *The functions* $I_i \in C[\Phi_i, R^n]$ *and* $(Id + I_i) : \Phi_i \to D$, $i = 1, 2, ...$.

H3. *For any point* $(t_0, x_0) \in R^+ \times D$ *and for each* $i = 1, 2, ...$, *the solution of the initial problem (5) exists and it is unique for* $t \geq t_0$.

H4. *The equalities* $\| a_i \| = 1$, $i = 1, 2, ...$ *are satisfied.*

H5. *The next inequalities are valid:*

$$(\langle a_i, (Id + I_{i-1})(x) \rangle - a_i) \langle a_i, f_i(t, x) \rangle < 0, (t, x) \in R^+ \times D,\ i = 1, 2, ...,$$

where $I_0(x) = 0$, $x \in D$.

H6. *There exist constants* $C_{\langle a_i, f_i\rangle} > 0$ *such that*

$$(\ \forall (t,x) \in R^+ \times D\ ) \Rightarrow \left\| \langle a_i, f_i(t,x) \rangle \right\| \geq C_{\langle a_i f_i \rangle}, i = 1,2,\dots .$$

H7. *There exist constants* $C_{a_i} > 0$ *such that*

$$(\forall x \in \Phi_i) \Rightarrow \left| \langle a_{i+1}, (Id + I_i)(x) \rangle - \alpha_{i+1} \right| \leq C_{a_{i+1}},\ i = 1,2,\dots .$$

H8. *The series* $\sum_{i=1}^{\infty} \frac{C_{a_{i+1}}}{C_{\langle a_{i+1}, f_{i+1} \rangle}}$ *are convergent.*

Theorem 1. *Let the conditions H1÷ H6 be fulfilled. Then the trajectory of problem (1), (2), (3), (4) meets each one of the hyperplanes* Φ_i, $i = 1,2,\dots$.

Proof. We will show that the trajectory of the considered problem meets the hyperplane Φ_1. From condition H5 it follows that one of the following two cases is satisfied:

Case 1. $(\langle a_1, x \rangle - \alpha_1) < 0$, $x \in D$ and $\langle a_1, f_1(t,x) \rangle > 0, (t,x) \in R^+ \times D$;

Case 2. $(\langle a_1, x \rangle - \alpha_1) > 0$, $x \in D$ and $\langle a_1, f_1(t,x) \rangle < 0$, $(t,x) \in R^+ \times D$.

Here, we will look at the second case. The first case is considered similarly. We introduce a function $\psi_1(t) = \langle a_1, x_1(t;t_0,x_0) \rangle - \alpha_1$, where $x_1(t;t_0,x_0)$ is a solution of problem (5) for $i = 1$. The function ψ_1 is defined for $t \in J(t_0, x_0, f_1) = [t_0, \infty)$. We have

$$\psi_1(t_0) = \langle a_1, x_1(t_0;t_0,x_0) \rangle - \alpha_1 = \langle a_1, x_0 \rangle - \alpha_1 > 0 .$$

According to condition H6, it is satisfied

$$\begin{aligned} \frac{d}{dt}\psi_1(t) &= \left\langle a_1, \frac{d}{dt} x_1(t;t_0,x_0) \right\rangle \\ &= \langle a_1,\ f_1(t, x_1(t;t_0,x_0)) \rangle \\ &= -\left| \langle a_1,\ f_1(t, x_1(t;t_0,x_0)) \rangle \right| \\ &\leq -C_{\langle a_1, f_1 \rangle} = -const < 0 . \end{aligned}$$

From the fact

$$\psi_1(t_0) > 0 \quad and \quad \frac{d}{dt}\psi_1(t) \leq -const < 0,\ t > t_0 ,$$

it follows that there exists a point $t_1 > t_0$ such that

$$\langle a_1, x_1(t_1;t_0,x_0) \rangle - \alpha_1 = \psi_1(t_1) = 0 .$$

This means that at the moment t_1, the trajectory $\gamma_1(t_0, x_0)$ meets the hyperplane Φ_1. Given that

$$\gamma(t_0, x_0) \equiv \gamma_1(t_0, x_0) \quad \text{for} \quad t_0 \leq t \leq t_1 ,$$

we conclude that the trajectory of problem (1), (2), (3), (4) also meets the hyperplane Φ_1 at the moment t_1.

Assume that the trajectory of investigated problem consistently meets the hyperplanes $\Phi_1, \Phi_2, \dots, \Phi_i$ at the moments $t_1, t_2, \dots, t_i$, respectively. It is fulfilled $t_1 < t_2 <, \dots, < t_i$. We will show that the trajectory $\gamma_{i+1}(t_0, x(t_i + 0; t_0, x_0))$ meets the hyperplane Φ_{i+1}, from which it follows that the same is true for the studied trajectory $\gamma(t_0, x_0)$. Again, taking into account condition H5, without loss of generality, we will suppose that the following inequalities are valid:

$$\langle a_{i+1},(Id+I_i)(x)\rangle-\alpha_{i+1}>0,\ x\in D \quad\text{and}\quad \langle a_{i+1}, f_{i+1}(t,x)\rangle<0,(t,x)\in R^+\times D\ . \tag{6}$$

We consider the function ψ_{i+1}, which is defined by

$$\psi_{i+1}(t)=\langle a_{i+1},x_{i+1}(t;t_i,x(t_i+0;t_0,x_0))\rangle,\quad t\ge t_i\,. \tag{7}$$

We have

$$\begin{aligned}\psi_{i+1}(t_i)&=\langle a_{i+1},x_{i+1}(t_i;t_i,x(t_i+0;t_0,x_0))\rangle-\alpha_{i+1}\\&=\langle a_{i+1},x(t_i+0;t_0,x_0)\rangle-\alpha_{i+1}\\&=\langle a_{i+1},x(t_i;t_0,x_0)+I_i(x(t_i;t_0,x_0))\rangle-\alpha_{i+1}\\&=\langle a_{i+1},(Id+I_i)(x(t_i;t_0,x_0))\rangle-\alpha_{i+1}>0\,.\end{aligned}$$

For $t>t_i$ it is satisfied

$$\begin{aligned}\frac{d}{dt}\psi_{i+1}(t)&=\langle a_{i+1}, f_{i+1}(t,x_{i+1}(t;t_0,x(t,t_0,x_0)))\rangle\\&=-\left|\langle a_{i+1}\ , f_{i+1}(t,x_{i+1}(t;t_0,x(t,t_0,x_0)))\rangle\right|\\&\le -C_{\langle a_{i+1}, f_{i+1}\rangle}=-const<0\,.\end{aligned}$$

Therefore, there exists a point $t_{i+1}>t_i$ such that

$$\psi_{i+1}(t_{i+1})=0 \iff \langle a_{i+1},x_{i+1}(t_{i+1};t_0,x(t_i+0;t_0,x_0))\rangle-\alpha_{I+1}=0\,.$$

The last equality shows that the trajectory $\gamma_{i+1}(t_0,x(t_i+0;t_0,x_0))$ meets the hyperplane Φ_{i+1} at the moment t_{i+1}. The same applies to the trajectory $\gamma(t_0,x_0)$.

The proof of the theorem follows by induction.

□

Theorem 2. *Let the conditions H1÷ H7 be fulfilled. Then the next estimates are valid*

$$t_{i+1}-t_i\le\frac{C_{a_{i+1}}}{C_{\langle a_{i+1}, f_{i+1}\rangle}},\ i=1,2,\ldots.$$

Proof. Let i be an arbitrary natural number. We consider the function ψ_{i+1}, which is defined by equality (7). Directly, we obtain the next equality

$$\psi_{i+1}(t)=\begin{cases}\langle a_{i+1},x(t_i+0;t_0,x_0)\rangle-\alpha_{i+1}\\ \qquad=\langle a_{i+1},x_i(t_i;t_0,x_0)+I_i(x_i(t_i;t_0,x_0))\rangle-\alpha_{i+1},\ t=t_i;\\ \langle a_{i+1},x(t;t_0,x_0)\rangle-\alpha_{i+1},\ t_i<t\le t_{i+1}.\end{cases}$$

Again, we suppose that the inequalities (6) are valid. Using condition H7, we receive

$$\begin{aligned}&\psi_{i+1}(t_{i+1})-\psi_{i+1}(t_i)\\&=\langle a_{i+1},x(t_{i+1};t_0,x_0)\rangle-\langle a_{i+1},x(t_i+0;t_0,x_0)\rangle\\&=-\langle a_{i+1},x_i(t_i;t_0,x_0)+I_i(x_i(t_i;t_0,x_0))\rangle+\alpha_{i+1}\\&=\left|\langle a_{i+1},(Id+I_i)(x_i(t_i;t_0,x_0))\rangle-\alpha_{i+1}\right|\\&\le C_{a_{i+1}}\,.\end{aligned} \tag{8}$$

On the other hand, using the conditions H6 and H4 consistently, we obtain

$$\psi_{i+1}(t_{i+1}) - \psi_{i+1}(t_i) \tag{9}$$

$$= \frac{d}{dt}\psi_{i+1}(t^*)(t_{i+1} - t_i)$$

$$= \frac{d}{dt}\left(\left\langle a_{i+1}, x(t^*; t_0, x_0)\right\rangle - \alpha_{i+1}\right).(t_{i+1} - t_i)$$

$$= \frac{d}{dt}\left(\left\langle a_{i+1}, x_{i+1}(t^*; t_0, x(t_i + 0, t_0, x_0))\right\rangle - \alpha_{i+1}\right).(t_{i+1} - t_i)$$

$$= \left\langle a_{i+1}, f_{i+1}\left(t^*, x_{i+1}(t^*; t_0, x(t_i + 0, t_0, x_0))\right)\right\rangle.(t_{i+1} - t_i)$$

$$\geq \|a_{i+1}\|.C_{\langle a_{i+1}, f_{i+1}\rangle}.(t_{i+1} - t_i)$$

$$= C_{\langle a_{i+1}, f_{i+1}\rangle}.(t_{i+1} - t_i),$$

where the point t^* satisfies the inequalities $t_i < t^* < t_{i+1}$. From (8) and (9) it follows the wanted estimate.

□

Theorem 3. *Let the conditions H1÷H8 be fulfilled. Then the solutions of system (1), (2), (3) die due to the impulsive effects.*

Proof. It is valid

$$J(t_0, x_0, f) = [t_0, t_1] \cup (t_1, t_2] \cup (t_2, t_3] \cup \ldots = [t_0, t^0), \text{ where}$$

$$t^0 = \lim_{i\to\infty} t_i$$

$$= t_1 + \lim_{i\to\infty}\left((t_2 - t_1) + (t_3 - t_2) + \ldots + (t_i - t_{i-1})\right)$$

$$= t_1 + \sum\nolimits_{i=1}^{\infty}(t_{i+1} - t_i)$$

$$= t_1 + \sum\nolimits_{i=1}^{\infty}\frac{C_{\varphi_{i+1}}}{C_{\langle grad\varphi_{i+1}, f_{i+1}\rangle}} < \infty.$$

□

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