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ON SOME (AOR) ITERATIVE ALGORITHMS FOR SOLVING SYSTEM OF LINEAR EQUATIONS

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Abstract. Some accelerated overrelaxation (AOR) iterative methods based on the Nekrassov–Mehmke procedure for finding solution of linear system of algebraic equations $Ax = b$ are given by the decomposition $A = T_m - E_m - F_m$, where T_m is a banded matrix of bandwidth $2m + 1$. We study the convergence of the new methods, based on the ideas given in [1], [2] and [3]. An interesting numerical example is presented.

Key words: solving linear system of equations, Nekrassov-Mehmke methods, Generalized Nekrassov-Mehmke methods, Successive Over Relaxation Generalized Nekrassov-Mehmke methods, Generalized Accelerated Over Relaxation methods, Symmetric Positive Definite (SPD) matrices, M-matrix
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1. Introduction

Let us consider the linear system $Ax - b = 0$, $(\det A \neq 0)$, or

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i = 0, \quad i = 1, 2, \ldots, n. \tag{1}$$

Suppose that the matrix A is strictly diagonally dominant (SDD), i.e. $|a_{ii}| > \sum_{j \neq i}^{n} |a_{ij}|$, $i = 1, 2, \ldots, n$.

In this paper we propose new iterative algorithms based on the classical methods of Nekrassov–Mehmke.

Using the Nekrassov–Mehmke iteration scheme, (or Gauss–Seidel scheme), see Nekrassov [4], Mehmke [5] and Nekrassov and Mehmke [6], the sequence of consecutive approximations x_i^k, is computed in this way:

$$(2) \qquad x_i^{k+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^k + \frac{b_i}{a_{ii}}, \qquad \begin{array}{l} i = 1, 2, \dots, n; \\ k = 0, 1, 2, \dots . \end{array}$$

Here after, we shall call the above scheme the *Nekrassov–Mehmke 1–method (NM1).* In a number of cases the success of the procedures of type (2) depends on the proper ordering of the equations (and x_i, $i = 1, \dots, n$) in system (1).

In spite of this fact the following modification of the Nekrassov–Mehmke method is known (see Faddeev D. and Faddeeva V. [7]):

$$(3) \qquad x_i^{k+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^k - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + \frac{b_i}{a_{ii}}, \qquad \begin{array}{l} i = n, n-1, \dots, 1; \\ k = 0, 1, 2, \dots . \end{array}$$

Here after, we shall call the above scheme the *Nekrassov–Mehmke 2–method (NM2).*

In [7] Faddeev D. and Faddeeva V. especially pointed out that of certain interest are such iteration processes in which cycles studied in two Nekrassov–Mehmke methods (NM1) and (NM2) are alternated.

The (NM2)-method does not possess better convergence in comparison with method (NM1).

But under circumstances, if matrix A is positive definite then the eigenvalues of matrix G in the matrix equations $x = Gx + t$ are real and this allows to apply different methods for improving rate of convergence, i.e. Abramov's technique [8].

Let $A = (a_{ij})$ be an $n \times n$ matrix and $T_m = (t_{ij})$ be a banded matrix of bandwidth $2m + 1$ defined as

$$t_{ij} = \begin{cases} a_{ij}, & |i - j| \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$T_m = \begin{pmatrix} a_{11} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1,1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{pmatrix},$$

$$E_m = \begin{pmatrix} & & \\ & & \\ -a_{m+2,1} & & \\ \vdots & \ddots & \\ -a_{n,1} & \cdots & -a_{n,n-m-1} \end{pmatrix},$$

$$F_m = \begin{pmatrix} & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & \ddots & \vdots \\ & & & -a_{n-m-1,n} \\ & & & \end{pmatrix}.$$

Applying the *Nekrassov–Mehmke method (NM1)* to the system $Ax = b$ with the decomposition $A = T_m - E_m - F_m$, i.e.

$$x^{k+1} = (T_m - E_m)^{-1} F_m x^k + (T_m - E_m)^{-1} b, \quad k = 0, 1, 2, \ldots . \tag{4}$$

Salkuyeh proved in [1] that the *generalized Nekrassov–Mehmke method* (GNM1) is convergent for any initial point x^0.

The following generalization of the (NM2) method–*generalized Nekrassov–Mehmke method* (GNM2) is proposed by Zaharieva, Kyurkchiev and Iliev in [9]:

$$x^{k+1} = (T_m - F_m)^{-1} E_m x^k + (T_m - F_m)^{-1} b, \quad k = 0, 1, 2, \ldots . \tag{5}$$

Let ω be a parameter such that the matrix $T_m - \omega E_m$ be nonsingular.

In [2] Salkuyeh considers the following *Successive Over Relaxation Generalized Nekrassov–Mehmke method* (GNM1) – (SORGNM1):

$$\begin{aligned} x^{k+1} &= (T_m - \omega E_m)^{-1} \big(\omega F_m + (1-\omega) T_m\big) x^k \\ &\quad + (T_m - \omega E_m)^{-1} \omega b, \qquad k = 0, 1, 2, \ldots . \end{aligned} \tag{6}$$

which is based on method (4) and proves the following convergence theorem for method (6).

Theorem A [2]. *Let A and T_m be symmetric positive definite (SPD) matrices. Then for every $0 < \omega < 2$, the method (6) converges.*

2. Main results

Let ω be a fixed parameter so that the matrix $T_m - \omega F_m$ be nonsingular. In this paper the following *Successive Over Relaxation Generalized Nekrassov–Mehmke method* (GNM2) – (SORGNM2) is proposed:

$$
\begin{aligned}
x^{k+1} &= (T_m - \omega F_m)^{-1}\big(\omega E_m + (1-\omega)T_m\big)x^k + (T_m - \omega F_m)^{-1}\omega b \\
&= Gx^k + (T_m - \omega F_m)^{-1}\omega b, \qquad\qquad k = 0,\, 1,\, 2,\, \ldots
\end{aligned} \tag{7}
$$

based on method (5).

We give a convergence theorem for method (7).

Theorem 1. *Let A and T_m be (SPD) matrices. Then for every $0 < \omega < 2$, the method (7) converges with any initial guess x^0.*

Proof. The proof follows the ideas given in [10] (see, Ostriwski-Reich's theorem) and [2].

Obviously $E_m = F_m^T$, since A is symmetric.

We prove that the matrix

$$S = \frac{1}{\omega} T_m - F_m$$

is nonsingular.

By contradiction, let S be singular. i.e. $Sx = 0$ for the nonzero vector x. In this case $x^T S x = 0$. Matrix A is an (SPD) matrix and

$$0 < x^T A x = x^T (T_m - F_m - F_m^T) x = x^T T_m x - 2x^T F_m x,$$

$$x^T F_m x < \frac{1}{2} x^T T_m x,$$

$$x^T S x = \frac{1}{\omega} x^T (T_m - \omega F_m) x = \frac{1}{\omega}(x^T T_m x - \omega x^T F_m x) > \frac{1}{\omega}\left(1 - \frac{\omega}{2}\right) x^T T_m x.$$

The matrix T_m is (SPD). Hence $x^T T_m x > 0$. On the other hand $1 - \dfrac{\omega}{2} > 0$. Therefore $x^T S x > 0$, which is a contradiction. We have

$$S + S^T - A = \frac{1}{\omega} T_m - F_m + \frac{1}{\omega} T_m - F_m^T - (T_m - F_m - F_m^T) = \left(\frac{2}{\omega} - 1\right) T_m.$$

The matrix T_m is (SPD), $\frac{2}{\omega} - 1 > 0$, then the matrix $S + S^T - A$ is (SPD). Let

$$R = A^{-1}(2S - A).$$

We show that if λ is an eigenvalue of R, then $Re\,\lambda > 0$. Let (λ, x) be an eigenpair of R. We have

$$A^{-1}(2S - A)x = \lambda x,$$

$$(2S - A)x = \lambda A x,$$

$$x^T(2S - A)x = \lambda x^T A x, \tag{8}$$

$$x^T(2S^T - A^T)x = x^T(2S^T - A)x = \overline{\lambda} x^T A x, \tag{9}$$

By adding the two sides of (8) and (9), we get

$$x^T(2S^T - A + 2S - A)x = (\lambda + \overline{\lambda}) x^T A x,$$

$$x^T(S^T - A + S)x = \frac{\lambda + \overline{\lambda}}{2} x^T A x = Re\,\lambda\, x^T A x.$$

Both A and $S + S^T - A$ are (SPD) matrices, hence we conclude that $Re\,\lambda > 0$.

It can be easily seen that $R + I$ is nonsingular. Therefore,

$$\begin{aligned}
(R - I)(R + I)^{-1} &= (A^{-1}(2S - A) - I)(A^{-1}(2S - A) + I)^{-1} \\
&= (2A^{-1}S - I - I)(2A^{-1}S - I + I)^{-1} \\
&= 2(A^{-1}S - I)\frac{1}{2}(A^{-1}S)^{-1} = I - S^{-1}A \\
&= I - \left(\frac{1}{\omega}T_m - F_m\right)^{-1}(T_m - E_m - F_m) \\
&= (T_m - \omega F_m)(T_m - \omega F_m)^{-1} \\
&\quad - \omega(T_m - \omega F_m)^{-1}(T_m - E_m - F_m) \\
&= (T_m - \omega F_m)^{-1}(T_m - \omega F_m - \omega T_m + \omega E_m + \omega F_m) \\
&= (T_m - \omega F_m)^{-1}\left((1 - \omega)T_m + \omega E_m\right) = G.
\end{aligned}$$

Let (μ, x) be an eigenpair of the matrix G.

Then

$$(R - I)(R + I)^{-1}x = \mu x.$$

By setting $z = (R + I)^{-1}x$, we see that $z \neq 0$. Hence,

$$x = (R + I)z,$$

$$(R - I)z = \mu(R + I)z,$$

$$(1 - \mu)Rz = (1 + \mu)z.$$

We have $\mu \neq 1$, since $z \neq 0$. Hence,

$$Rz = \frac{1 + \mu}{1 - \mu} z.$$

This relation shows that $\lambda = \dfrac{1 + \mu}{1 - \mu}$ is an eigenvalue of R. As a result we have $\mu = \dfrac{\lambda - 1}{\lambda + 1}$ and

$$|\mu|^2 = \mu\,\overline{\mu} = \frac{|\lambda|^2 + 1 - 2Re\,\lambda}{|\lambda|^2 + 1 + 2Re\,\lambda}\,.$$

Having in mind that $Re\,\lambda > 0$, we conclude that

$$|\mu| < 1 \;\; \rightarrow \rho(G) < 1.$$

This completes the proof.

□

For other results, see [11], [13], [12], [14] and [15].

Now, similar to the classical (AOR) method [16] its generalized version is defined as following (see, Salkuyeh in [3]) *Generalized Accelerated Over Relaxation Method* – (G_{AOR}), based on the Nekrassov–Mehmke method (GNM1):

$$\begin{aligned} x^{k+1} = {} & (T_m - \gamma E_m)^{-1}\left((1 - \omega)T_m + (\omega - \gamma)F_m + \omega F_m\right)x^k \\ & + \omega(T_m - \gamma E_m)^{-1}b, \qquad k = 0,\, 1,\, 2,\, \dots\,, \end{aligned} \tag{10}$$

based on method (6), where $0 \leq \gamma < \omega \leq 1$.

Let $G^{(m)}_{AOR}(\gamma,\omega)$ be the iteration matrix of the method (10), i.e.

$$G^{(m)}_{AOR}(\gamma,\omega) = (T_m - \gamma E_m)^{-1}\left((1-\omega)T_m + (\omega-\gamma)F_m + \omega F_m\right).$$

Procedure (10) is valid in the case where A is an M - matrix.

A matrix $A = (a_{ij})$ is said to be an M - matrix, if $a_{ii} > 0$ for $i = 1, 2, \dots, n$, $a_{ij} \le 0$ for $i \ne j$, A is nonsingular and $A^{-1} \ge 0$.

In [3] Salkuyeh proves the following convergence theorem for method (10).

Theorem B [3]. *If A is an M matrix and $0 \le \gamma < \omega \le 1$ with $\omega \ne 0$, then the method* (10) *is convergent, i.e.*

$$\rho\left(G^{(m)}_{AOR}(\gamma,\omega)\right) < 1.$$

We propose the following method *Generalized Accelerated Over Relaxation Method* – (G^N_{AOR}), based on Nekrassov–Mehmke method (GNM2):

$$\begin{aligned} x^{k+1} &= (T_m - \gamma F_m)^{-1}\left((1-\omega)T_m + (\omega-\gamma)F_m + \omega E_m\right)x^k \\ &\quad + \omega(T_m - \gamma F_m)^{-1}b, \qquad\qquad k = 0, 1, 2, \dots, \end{aligned} \tag{11}$$

based on method (7), where $0 \le \gamma < \omega \le 1$.

Let $G^{N,(m)}_{AOR}(\gamma,\omega)$ be the iteration matrix of method (11), i.e.

$$G^{N,(m)}_{AOR}(\gamma,\omega) = (T_m - \gamma E_m)^{-1}\left((1-\omega)T_m + (\omega-\gamma)F_m + \omega F_m\right).$$

We give a convergence theorem for method (11).

Theorem 2. *If A is an M matrix and $0 \le \gamma < \omega \le 1$ with $\omega \ne 0$, then method* (11) *is convergent, i.e.*

$$\rho\left(G^{N,(m)}_{AOR}(\gamma,\omega)\right) < 1.$$

Proof. The proof follows the ideas given in [3]. For the G^N_{AOR} method we have $A_m = M^N_m - N^N_m$, where

$$M^N_m = T_m - \gamma F_m,$$

$$N^N_m = (1-\omega)T_m + (\omega-\gamma)F_m + \omega E_m,$$

and $A \leq M_m^N$, $\left(M_m^N\right)^{-1} \geq 0$ and M_m^N is an M - matrix.

On the other hand (see, [18], [3]),

$$\rho\left(\left(T_m\right)^{-1} F_m\right) < 1.$$

For $0 \leq \gamma \leq 1$,

$$\rho\left(\gamma\left(T_m\right)^{-1} F_m\right) < 1$$

and therefore,

$$\begin{aligned}\left(M_m^N\right)^{-1} N_m^N &= \left(T_m - \gamma F_m\right)^{-1}\left[(1-\omega)T_m + (\omega-\gamma)F_m + \omega E_m\right] \\ &= \left(I - \gamma T_m^{-1} F_m\right)^{-1}\left[(1-\omega)I + (\omega-\gamma)T_m^{-1}F_m + \omega T_m^{-1}E_m\right] \\ &\geq 0.\end{aligned}$$

We note that $\omega A = M_m^N - N_m^N$ is a weak regular splitting of ωA. From the result by Wang and Song [18], we observe that

$$\rho\left(\left(M_m^N\right)^{-1} N_m^N\right) = \rho\left(G_{AOR}^{N,(m)}(\gamma,\omega)\right) < 1$$

and this completes the proof.

□

For other results, see [17], [18], [19], [20] and [21].

3. Numerical example

Consider the M - matrix (example by Salkueh [3]):

$$A = \begin{pmatrix} 4 & -2 & -1 & -2 \\ -1 & 5 & -5 & -1 \\ -2 & -1 & 9 & -1 \\ -1 & -1 & -1 & 5 \end{pmatrix}.$$

Let $m = 1$, $\gamma = 0.5$, $\omega = 0.9$. For method (11) we have

$$M_1^N = \begin{pmatrix} 4 & -2 & -0.5 & -1 \\ -1 & 5 & -5 & -0.5 \\ 0 & -1 & 9 & -1 \\ 0 & 0 & -1 & 5 \end{pmatrix}$$

$$N_1^N = \begin{pmatrix} 0.4 & -0.2 & 0.4 & 0.8 \\ -0.1 & 0.5 & -0.5 & 0.4 \\ 1.8 & -0.1 & 0.9 & -0.1 \\ 0.9 & 0.9 & -0.1 & 0.5 \end{pmatrix}$$

$$\left(M_1^N\right)^{-1} = \begin{pmatrix} 0.283321 & 0.133285 & 0.0997815 & 0.089949 \\ 0.0640932 & 0.256373 & 0.153678 & 0.0691916 \\ 0.00728332 & 0.0291333 & 0.1311 & 0.0305899 \\ 0.00145666 & 0.00582666 & 0.02622 & 0.206118 \end{pmatrix}$$

$$\left(M_1^N\right)^{-1} N_1^N = \begin{pmatrix} 0.360561 & 0.0809541 & 0.127495 & 0.314967 \\ 0.338893 & 0.162272 & 0.028842 & 0.173052 \\ 0.263511 & 0.027531 & 0.103277 & 0.019665 \\ 0.232702 & 0.185506 & 0.000655499 & 0.103933 \end{pmatrix}$$

For the eigenvalues of the matrix $\left(M_1^N\right)^{-1} N_1^N$ we have:

$$\begin{aligned} &0.132076, \\ &0.701942, \\ &-0.0519868 + 0.0406157\, I, \\ &-0.0519868 - 0.0406157\, I, \end{aligned}$$

and for the spectral radius of $\left(M_1^N\right)^{-1} N_1^N$:

$$\rho\left(G_{AOR}^{N,(1)}(0.5, 0.9)\right) = 0.701942 < 1.$$

The result shows that Theorem 2 holds true.

4. Concluding remarks

Remark 1. We shall point out that in the case of symmetry, methods (6) and (7) are equivalent.

Remark 2. This is not the case, when A is an M - matrix. For method (10) Salkueh proved that

$$\rho\left(G_{AOR}^{(1)}(0.5, 0.9)\right) = 0.677571 < 1.$$

In our case,

$$\rho\left(G^{N,(1)}_{AOR}(0.5,0.9)\right) = 0.701942 < 1.$$

For $m = 2$ (see, Salkueh),

$$\rho\left(G^{(2)}_{AOR}(0.5,0.9)\right) = 0.5053 < 1.$$

From (11) we have

$$\rho\left(G^{N,(2)}_{AOR}(0.5,0.9)\right) = 0.495377 < 1,$$

i.e.

$$\rho\left(G^{N,(2)}_{AOR}(0.5,0.9)\right) < \rho\left(G^{(2)}_{AOR}(0.5,0.9)\right),$$

which shows that the method (11) has its right of existence.

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ВЪРХУ НЯКОИ ИТЕРАЦИОННИ АЛГОРИТМИ ЗА РЕШАВАНЕ НА СИСТЕМИ ОТ ЛИНЕЙНИ УРАВНЕНИЯ

Десислава Захариева, Анна Малинова

Резюме. Изследвани са някои итерационни методи за числено решаване на системи линейни уравнения, базиращи се на метода на Nekrassov–Mehmke (обратен ход) от тип горна релаксация с два параметъра ω, γ приложени за лентови матрици с ширина $2m + 1$. Доказани са теореми за сходимост и е показано с подходящ пример, че в случая на M - матрици, методите имат право на съществуване.