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EVENTUAL φ_0-STABILITY OF DIFFERENTIAL SYSTEMS IN TERMS OF TWO MEASURES BY PERTURBING LYAPUNOV FUNCTIONS

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Abstract. The notation of eventual φ_0-stability of nonlinear systems of ordinary differential equations in terms of two measures is introduced. Our technique depends on Lyapunov direct method. Perturbing cone-valued Lyapunov functions have been applied and comparison scalar ordinary differential equations have been employed.

Key words: stability in terms of two measures, Lyapunov functions, ordinary differential equations
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1. Introduction

Lakshmikantham and Leela [7] initiated the development of theory of differential inequalities through cones and cone-valued Lyapunov function method. Furthermore the same authors in [5], [8] used the comparison principal to improve and extend different types of stability, say eventual stability and (h_0, h)-stability for the differential systems.

Akpan and Akinyele [1] discussed φ_0-stability notions of the comparison system. Hristova [4] introduced a new notion of stability called φ_0-stability in terms of two measures for the differential systems.

In [2] the authors extended the notion of (h_0, h)-stability to the so-called (h_0, h)-eventual stability.

In the present paper eventual φ_0-stability of ordinary differential systems in terms of two measures is studied. Cone-valued Lyapunov functions are employed as well as comparison results.

2. Preliminary notes and definitions

Consider the system of nonlinear differential equations

$$\dot{x} = f(t, x) \quad \text{for } t \geq t_0, \tag{1}$$

where $x \in R^n$ and $f \in C[R^+ \times R^n, R^n]$.

We denote by $x(t; t_0, x_0)$ the solution of the system (1) with initial condition $x(t_0; t_0, x_0) = x_0$.

Consider the following sets

$$K = \{a \in C[R^+, R^+] : \ a(s) \text{ is strictly increasing and } a(0) = 0\};$$
$$G = \{H \in C[R^+ \times R^+, R^+] : \ \inf_{s \in R^+} H(t, s) = 0 \text{ for each } t \in R^+\};$$
$$Z = \{\sigma \in C[R^+, R^+] : \sigma(t) \text{ is strictly decreasing and } \sigma(t) \to 0 \text{ as } t \to \infty\}.$$

Definition 1. ([1]). *A proper set $\mathcal{K} \subset R^n$ is called a cone if*

(i) $\lambda\mathcal{K} \subset \mathcal{K}, \ \lambda > 0;$
(ii) $\mathcal{K} + \mathcal{K} \subset \mathcal{K};$
(iii) $\overline{\mathcal{K}} = \mathcal{K};$
(iv) $\mathcal{K}^0 \neq \varnothing;$
(v) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}.$

The set

$$\mathcal{K}^* = \{\varphi \in R^n : (\varphi, x) \geq 0, \ x \in \mathcal{K}\}$$

is called a joint cone, if $\mathcal{K}^*$ is a cone.

Let ρ be a positive constant, $\varphi_0 \in \mathcal{K}$ and $H \in G$. Define the sets:

$$\tilde{S}(H, \rho, \varphi_0) = \{(t, x) \in R^+ \times \mathcal{K} : H(t, (\varphi_0, x)) < \rho\};$$
$$\tilde{S}^c(H, \rho, \varphi_0) = \{(t, x) \in R^+ \times \mathcal{K} : H(t, (\varphi_0, x)) \geq \rho\}.$$

We will study eventual φ_0-stability in terms of two measures of ordinary differential systems. In the case when cone-valued Lyapunov functions are applied both measures are from the set G. In this case we will introduce the definition of a new type of stability, that combines the ideas of eventual stability in terms of two measures of ordinary differential equations ([2]) and φ_0-stability ([1]).

Definition 2. *Let $\varphi_0 \in \mathcal{K}^*$, H, $H_0 \in G$. The system (1) is said to be:*

$(S1)$ (H_0, H)*-eventually uniformly* φ_0*-stable if for every* $\varepsilon > 0$ *and for any* $t_0 \in R^+$, *there exist a* $\delta = \delta(\varepsilon) > 0$ *and a* $\tau = \tau(\varepsilon) > 0$ *such that*

$$H_0\big(t_0, (\varphi_0, x_0)\big) \le \delta \text{ implies } H\big(t, (\varphi_0, x(t;\, t_0, x_0))\big) < \varepsilon \text{ for } t \ge t_0 \ge \tau(\varepsilon),$$

where $x(t;\, t_0, x_0)$ *is any solution of system (1);*

$(S2)$ (H_0, H)*-eventually quasi-uniformly asymptotically* φ_0*-stable, if for every* $\varepsilon > 0$ *and for any* $t_0 \in R^+$, *there exist positive numbers* δ_0, τ_0 *and* $T = T(\varepsilon)$ *such that*

$$H_0(t_0, (\varphi_0, x_0)) \le \delta_0 \text{ implies } H\big(t, (\varphi_0, x(t;\, t_0, x_0))\big) < \varepsilon \text{ for } t \ge t_0 + T, \quad t_0 \ge \tau_0;$$

$(S3)$ (H_0, H)*-eventually uniformly asymptotically* φ_0*-stable if* $(S1)$ *and* $(S2)$ *hold together.*

In our further investigations we will use two different comparison scalar ordinary differential equations:

$$\dot{u} = g_1(t, u), \quad t \ge t_0, \tag{2}$$

and

$$\dot{v} = g_2(t, v), \quad t \ge t_0, \tag{3}$$

where $u,\ v \in R$, $g_i(t, 0) \equiv 0$ $(i = 1, 2)$.

We will use some properties of the functions from the class G.

Definition 3. ([3]). *Let* H, $H_0 \in G$. *Then we say that a function* $H_0(t, x)$ *is uniformly finer than a function* $H(t, x)$, *if there exist a constant* $\delta > 0$ *and a function* $a \in K$ *such that*

$$H_0(t, x) < \delta \quad \text{implies} \quad H(t, x) \le a\big(H_0(t, x)\big).$$

We will use the following class of functions:

Definition 5. ([4]). *We will say that the function* $V(t, x) : R^+ \times R^n \to \mathcal{K}$ *belongs to the class* L, *if:*

1. $V(t, x)$ *is continuous function for any* $t \in R^+$ *and* $x \in \mathcal{K}$;
2. *Function* $V(t, x)$ *is component-wisely Lipshitz in* x *relatively to* $\mathcal{K}$.

Definition 4. ([4]). *Let* $\varphi_0 \in \mathcal{K}^*$, $H \in G$. *Function* $V(t, x) \in L$ *is said to be* φ_0*-strongly* H*-decrescent, if there exist a constant* $\delta > 0$ *and a function* $a \in K$ *such that the inequality*

$$H(t, (\varphi_0, x)) < \delta \quad \text{implies} \quad (\varphi_0, V(t, x)) \le a\big(H(t, (\varphi_0, x))\big).$$

Let $t \geq t_0$, $x \in R^n$. We define the derivative of the function $V(t,x) \in L$ along the trajectory of the solution of (1) as follows

$$D^+_{(1)}V(t,x) = \lim_{\varepsilon \to 0^+} \sup(1/\varepsilon)\Big\{V\big(t+\varepsilon,\, x+\varepsilon f(t,x)\big) - V(t,x)\Big\}.$$

In the further investigations we will use the following comparison result:

Lemma 1. (Theorem 1.4.1., 5). *Let $E \subset R \times R$ be an open set and:*
1. Function $g_1 \in C[E,R]$.
2. Function $m \in C[[t_0, t_0+a) \times R \cap E, R]$ satisfies the inequalities

$$\dot{m} \leq g_1(t,m), \quad t \in [t_0, t_0+a), \quad m(t_0) \leq u_0.$$

3. Function $r^*(t) = r^*(t;\, t_0, u_0)$ is the maximal solution of (2) through the point (t_0, u_0), defined for $t \in [t_0, t_0+a)$.
Then

$$m(t) \leq r^*(t), \quad t \in [t_0, t_0+a).$$

3. Main results

We will obtain sufficient conditions for eventual uniform φ_0-stability and eventual uniform asymptotic φ_0-stability of system of ordinary differential equations in terms of two measures. We will employ two different types of Lyapunov functions from the class L. The proof is based on the second method of Lyapunov with perturbing Lyapunov functions combined with comparison results for scalar ordinary differential equations.

Theorem 1. *Let the following conditions be fulfilled:*
1. Function $f \in C[R^+\times R^n, \mathcal{K}]$.
2. Function H_0, $H \in G$, H_0 is uniformly finer than H.
3. Function $\varphi_0 \in \mathcal{K}^*$.
4. There exists a function $V_1 \in L$ that is φ_0-strongly H_0-decrescent and

$$(i)\ \big(\varphi_0, D^+_{(1)}V_1(t,x)\big) \leq g_1\big(t, (\varphi_0, V_1(t,x))\big) \text{ for } (t,x) \in \tilde{S}(H,\rho,\varphi_0),$$

where $g_1(t,u) \in C[R^+\times R, R]$, $g_1(t,0) \equiv 0$ and $\rho = const > 0$.
5. For any number $\mu > 0$ there exists a function $V_2^{(\mu)} \in L$ and

$$(ii)\ b\big(H(t,(\varphi_0,x))\big) \leq \big(\varphi_0, V_2^{(\mu)}(t,x)\big) \leq a\big(H_0(t,(\varphi_0,x))\big)$$

for $(t,x) \in \tilde{S}(H,\rho,\varphi_0) \cap \tilde{S}^c(H_0,\mu,\varphi_0)$, where a, $b \in K$ and $b(u) \to \infty$ as $u \to \infty$.

(iii) for any point $(t,x) \in \tilde{S}(H,\rho,\varphi_0) \cap \tilde{S}^c(H_0,\mu,\varphi_0)$ *the inequality*

$$\Big(\varphi_0, \Big\{D^+_{(1)}V_1(t,x) + D^+_{(1)}V_2^{(\mu)}(t,x)\Big\}\Big) \le g_2\Big(t, \big(\varphi_0, V_1(t,x) + V_2^{(\mu)}(t,x)\big)\Big)$$

holds, where $g_2 \in C[R^+\times R, R]$, $g_2(t,0) \equiv 0$;

6. The conditions 2, 4 and 5 are satisfied for $0 < r < H(t,(\varphi_0,x))$ *and* $t \ge \theta(r)$, *where* $\theta \in Z$ *for* $0 < r < \rho$.

7. For any initial point $(t_0,x_0) \in R^+\times R^n$ *the solutions of the system (1) exist on* $[t_0,\infty)$.

8. For any initial point $(t_0,u_0) \in R^+\times R$ *the solutions of the scalar differential equations (2) and (3) exist on* $[t_0,\infty)$.

9. The zero solution of the scalar differential equation (2) is uniformly stable.

10. The zero solution of the scalar differential equation (3) is eventually uniformly stable.

Then the system of differential equations (1) is (H_0,H)*-eventually uniformly* φ_0*-stable.*

Proof. Since the function $V_1(t,x)$ is φ_0-strongly H_0-decrescent, there exist a constant $\rho_1 \in (0,\rho)$ and a function $\Psi_1 \in K$ such that

(4) $$H_0(t,(\varphi_0,x)) < \rho_1 \text{ implies } (\varphi_0, V_1(t,x)) \le \Psi_1\big(H_0(t,(\varphi_0,x))\big).$$

Since $H_0(t,x)$ is uniformly finer than $H(t,x)$, there exist a constant $\rho_0 \in (0,\rho_1)$ and a function $\Psi_2 \in K$ such that

(5) $$H_0(t,x) < \rho_0 \text{ implies } H(t,x) \le \Psi_2\big(H_0(t,x)\big),$$

where $\Psi_2(\rho_0) < \rho_1$.

Let $\varepsilon > 0$ be a positive number such that $\varepsilon < \rho$ and $t_0 \in R^+$ be a fixed number.

Since the zero solution of (3) is eventually uniformly stable, then given $b(\varepsilon) \in K$, there exist $\tau_1 = \tau_1(\varepsilon) > 0$ and $\delta_1(\varepsilon) > 0$ such that the inequality

(6) $$|v_0| < \delta_1 \text{ implies } |v(t;\, t_0, v_0)| < b(\varepsilon), \qquad t \ge t_0 \ge \tau_1(\varepsilon),$$

where $v(t;\, t_0, v_0)$ is any solution of (3) with an initial condition $v(t_0) = v_0$.

Since the functions $a \in K$ and $\Psi_2 \in K$ we can find $\delta_2 = \delta_2(\varepsilon) > 0$, $\delta_2 < \rho_0$, such that the inequalities

(7) $$a(\delta_2) < \frac{\delta_1}{2}, \qquad \Psi_2(\delta_2) < \varepsilon_1$$

hold.

Since the zero solution of (2) is uniformly stable, then given $\delta_1/2 > 0$ and $t_0 \in R^+$, there exists a $\delta_3 = \delta_3(\varepsilon) > 0$ such that

$$|u_0| < \delta_3 \ \text{ implies } \ |u(t;\, t_0, u_0)| < \frac{\delta_1}{2}\,, \quad t \geq t_0, \tag{8}$$

where $u(t;\, t_0, u_0)$ is any solution of (2) with an initial condition $u(t_0) = u_0$.

Since the function $\Psi_1 \in K$ there exists $\delta_4 = \delta_4(\delta_3) = \delta_4(\varepsilon) > 0$ such that for $|u| < \delta_4$ the inequality

$$\Psi_1(u) < \delta_3 \tag{9}$$

holds.

We choose $u_0 = \big(\varphi_0, V_1(t_0, x_0)\big)$. From inequalities (4) and (8) follows that there exists $\delta_5 = \delta_5(\varepsilon) > 0$, $\delta_5 < \min(\delta_4, \rho_1)$, such that $H_0\big(t_0, (\varphi_0, x_0)\big) < \delta_5$ implies $\big(\varphi_0, V_1(t_0, x_0)\big) \leq \Psi_1\big(H_0(t_0, (\varphi_0, x_0))\big) < \delta_3$. We set $\delta = \min(\delta_2, \delta_5)$ and now let the point $x_0 \in R^n$ be such that

$$H_0\big(t_0, (\varphi_0, x_0)\big) < \delta. \tag{10}$$

From inequalities (5) and (9) we get

$$H\big(t_0, (\varphi_0, x_0)\big) \leq \Psi_2\big(H_0(t_0, (\varphi_0, x_0))\big) \leq \Psi_2(\delta) < \Psi_2(\delta_2) < \varepsilon. \tag{11}$$

Going through as in the proof of Theorem 3.14.1 of [5], we define $\tau_2(\varepsilon) = \theta(\delta(\varepsilon))$, and let $\tau = \tau(\varepsilon) = \max\big[\tau_1(\varepsilon), \tau_2(\varepsilon)\big]$.

To prove the theorem, it must be shown that

$$H_0\big(t_0, (\varphi_0, x_0)\big) < \delta \ \text{ implies } \ H\big(t, (\varphi_0, x(t;\, t_0, x_0))\big) < \varepsilon, \quad t \geq t_0 \geq \tau(\varepsilon). \tag{12}$$

Suppose this is false. Therefore, there exists a point $t^* > t_0$ such that for $t \in [t_0, t^*)$ and $t_0 \geq \tau(\varepsilon)$ the following inequalities are valid

$$H(t^*, (\varphi_0, x(t^*;\, t_0, x_0))) \geq \varepsilon, \quad H(t, (\varphi_0, x(t;\, t_0, x_0))) < \varepsilon. \tag{13}$$

From the continuity of the solution $x(t;\, t_0, x_0)$ at point t^* it follows that $H\big(t^*, (\varphi_0, x(t^*;\, t_0, x_0))\big) = \varepsilon$.

Define $x(s) = x(s;\, t_0, x_0)$, $s \in [t_0, t^*]$.

If we assume that $H_0\big(t^*, (\varphi_0, x(t^*))\big) \leq \delta_2 < \rho$, then from the choice of δ_2 follows

$$H\big(t^*, (\varphi_0, x(t^*))\big) \leq \Psi_2\big(H_0(t^*, (\varphi_0, x(t^*)))\big) \leq \Psi_2(\delta_2) < \varepsilon,$$

which contradicts (13).

Therefore,

$$H_0\big(t^*,(\varphi_0,x(t^*))\big) > \delta_2, \quad H_0\big(t_0,(\varphi_0,x_0)\big) < \delta \le \delta_2. \tag{14}$$

From inequalities (14) follows that there exists a point $t_0^* \in (t_0,t^*)$ such that $\delta_2 = H_0\big(t_0^*,(\varphi_0,x(t_0^*))\big)$ and

$$(t,x(t)) \in \tilde{S}(H,\varepsilon,\varphi_0) \cap \tilde{S}^c(H_0,\delta_2,\varphi_0), \quad t \in [t_0^*,t^*). \tag{15}$$

Let $r_1(t;\, t_0,u_0)$ be the maximal solution of differential equation (2) where $u_0 = \big(\varphi_0, V_1(t_0,x_0)\big)$.

Define the function $p(t) = \big(\varphi_0, V_1(t,x(t))\big)$ for $t \in [t_0^*,t^*]$. Then

$$\begin{aligned} D^+p(t) &= \lim_{\varepsilon\to 0^+} \sup(1/\varepsilon)\{p(t+\varepsilon)-p(t)\} \\ &= \lim_{\varepsilon\to 0^+} \sup(1/\varepsilon)\Big\{\big(\varphi_0, V_1(t+\varepsilon,x(t+\varepsilon))\big) - \big(\varphi_0, V_1(t,x(t))\big)\Big\} \\ &= \Big(\varphi_0, \lim_{\varepsilon\to 0^+} \sup(1/\varepsilon)\Big\{V_1(t+\varepsilon,x(t+\varepsilon)) - V_1(t,x(t))\Big\}\Big) \\ &= \big(\varphi_0, D^+_{(1)}V_1(t,x(t))\big) \le g_1\big(t,(\varphi_0,V_1(t,x(t)))\big) = g_1(t,p(t)). \end{aligned} \tag{16}$$

According to Lemma 1 we obtain

$$p(s) = \big(\varphi_0, V_1(s,x(s))\big) \le r_1(s;\, t_0,u_0), \quad s \in [t_0,t^*]. \tag{17}$$

Condition (10) for the point x_0 and inequalities (4), (8) and (17) imply that

$$\big(\varphi_0, V_1(t_0^*,x(t_0^*))\big) < \frac{\delta_1}{2}\,. \tag{18}$$

Consider the function $V_2^{(\delta_2)}(t,x)$ that is defined in condition 5 of Theorem 1 and define the function

$$m(t,x) = V_1(t,x) + V_2^{(\delta_2)}(t,x), \quad t \ge t_0^*. \tag{19}$$

From inequality (7) and condition (iii) of Theorem 1 follows that

$$\big(\varphi_0, V_2^{(\delta_2)}(t_0^*,x(t_0^*))\big) < a\big(H_0(t_0^*,(\varphi_0,x(t_0^*)))\big) = a(\delta_2) < \frac{\delta_1}{2}\,. \tag{20}$$

From inequalities (18) and (20) we obtain

$$\big(\varphi_0, m(t_0^*,x(t_0^*))\big) < \delta_1. \tag{21}$$

Define the function $q(t) = \big(\varphi_0, m(t, x(t))\big)$ for $t \geq t_0^*$.

Let $t \in [t_0^*, t^*]$. Then using inclusion (15) and condition (iii) of Theorem 1 we obtain

$$(22) \qquad D^+ q(t) = \Big(\varphi_0, \Big\{D^+_{(1)} V_1(t, x(t)) + D^+_{(1)} V^{(\delta_2)}_2(t, x(t))\Big\}\Big) \leq g_2(t, q(t)).$$

According to inequality (22) we get

$$(23) \qquad q(t) = \big(\varphi_0, m(t, x(t;\, t_0, x_0))\big) \leq r^*(t;\, t_0^*, v_0^*) \quad \text{for } t \in [t_0^*, t^*],$$

where $r^*(t;\, t_0^*, v_0^*)$ is the maximal solution of (3) through the point (t_0^*, v_0^*), $v_0^* = \big(\varphi_0, m(t_0^*, x(t_0^*;\, t_0, x_0))\big)$.

From inequality (21) follows that $|v_0^*| < \delta_1$ and therefore according to inequality (6)

$$(24) \qquad r^*(t;\, t_0^*, v_0^*) < b(\varepsilon), \quad t \geq t_0^*.$$

From inequalities (23) and (24), the choice of the point t^*, and condition (iii) of Theorem 1 we obtain

$$b(\varepsilon) > r^*(t^*;\, t_0^*, v_0^*) \geq \big(\varphi_0, m(t^*, x(t^*;\, t_0, x_0))\big) \geq$$
$$\geq \big(\varphi_0, V^{(\delta_2)}_2(t^*, x(t^*;\, t_0, x_0))\big) \geq b\big(H(t^*, (\varphi_0, x(t^*;\, t_0, x_0)))\big) = b(\varepsilon).$$

The obtained contradiction proves the validity of inequality (12).

Inequality (12) proves the (H_0, H)-eventual uniform φ_0-stability of the considered system of differential equations. □

Theorem 2 *Let the conditions 1–5 of Theorem 1 be satisfied and let the following equality $g_2(t, v) = 0$ hold.*

Assume that

$$(H_1) \quad \Big(\varphi_0, \Big\{D^+_{(1)} V_1(t, x) + D^+_{(1)} V^{(\mu)}_2(t, x)\Big\}\Big) \leq -c\big(H(t, (\varphi_0, x))\big)$$

for $0 < r < H(t, (\varphi_0, x))$ and $t \geq \theta(r)$, where $c \in K$, $\theta \in Z$ for $0 < r < \rho$ and $(t, x) \in \tilde{S}(H, \rho, \varphi_0) \cap \tilde{S}^c(H_0, \mu, \varphi_0)$;

$$(H_2) \quad \big(\varphi_0, V_1(t, x) + V^{(\mu)}_2(t, x)\big) \leq \beta\big(H(t, (\varphi_0, x))\big)$$

for $0 < r < H(t, (\varphi_0, x))$ and $t \geq \theta(r)$, where $\beta \in K$ and

$$(t, x) \in \tilde{S}(H, \rho, \varphi_0) \cap \tilde{S}^c(H_0, \mu, \varphi_0).$$

Then the differential system (1) is (H_0, H)-eventually uniformly asymptotically φ_0-stable.

Proof. Since (H_1) implies that

$$\Big(\varphi_0, \Big\{D^+_{(1)}V_1(t,x)+D^+_{(1)}V^{(\mu)}_2(t,x)\Big\}\Big) \le 0 \text{ for } (t,x) \in \tilde{S}(H,\rho,\varphi_0)\cap\tilde{S}^c(H_0,\mu,\varphi_0).$$

The conditions $1-6$ of Theorem 1 and hypotheses (H_1) and (H_2), yield by Theorem 1, that the differential system (1) is (H_0,H)-eventually uniformly φ_0-stable. Let $\varepsilon > 0$ be given. Following [5], choose

$$\delta_0 = \delta(\rho), \quad \tau_0 = \tau(\rho) \text{ and } T(\varepsilon) = \tau(\varepsilon) + \beta(\rho)/c[\delta(\varepsilon)]. \tag{25}$$

To prove the Theorem 2, it must be shown that $H_0(t,(\varphi_0,x)) < \delta_0$ implies $H\big(t^*,(\varphi_0,x(t^*;\, t_0,x_0))\big) < \delta(\varepsilon)$, $t^* \in [t_0+\tau(\varepsilon), t_0+T(\varepsilon)]$.

Suppose that this is false. Then

$$\delta(\varepsilon) \le H(t,(\varphi_0,x(t))) < \rho, \quad t \in [t_0+\tau(\varepsilon), t_0+T(\varepsilon)]. \tag{26}$$

Let

$$m(t,x) = V_1(t,x) + V^{(\mu)}_2(t,x), \quad t \in [t_0+\tau(\varepsilon), t_0+T(\varepsilon)]. \tag{27}$$

By using the inequality (26) and the condition (H_1) of Theorem 2, for $t \in [t_0+\tau(\varepsilon), t_0+T(\varepsilon)]$ we get

$$D^+q(t) = \{\varphi_0, D^+m(t)\} \le -c\big(H(t,(\varphi_0,x(t)))\big) \le -c[\delta(\varepsilon)]. \tag{28}$$

By integrating (28) from $t_0+\tau(\varepsilon)$ to $t_0+T(\varepsilon)$, we obtain

$$\big\{\varphi_0, m(t_0+T(\varepsilon))\big\} \le \big\{\varphi_0, m(t_0+\tau(\varepsilon))\big\} - c[\delta(\varepsilon)][T(\varepsilon)-\tau(\varepsilon)]. \tag{29}$$

Then from condition (ii) of Theorem 1, hypothesis (H_2) of Theorem 2, (25), (26) and (29), we obtain

$$\begin{aligned} 0 &< b[\delta(\varepsilon)] \\ &\le \{\varphi_0, m(t_0+T(\varepsilon))\} \\ &\le \beta\Big(H\big(t_0+\tau(\varepsilon),(\varphi_0,x(t_0+\tau(\varepsilon);\, t_0,x_0))\big)\Big) - c[\delta(\varepsilon)]\frac{\beta(\rho)}{c[\delta(\varepsilon)]} \\ &\le \beta(\rho) - \beta(\rho) = 0, \end{aligned}$$

which is a contradiction. Then there exists a $t^* \in [t_0+\tau(\varepsilon), t_0+T(\varepsilon)]$ such that $H\big(t^*,(\varphi_0,x(t^*;\, t_0,x_0))\big) < \delta(\varepsilon)$.

Therefore, the system (1) is (H_0,H)-eventually uniformly asymptotically φ_0-stable.

□

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ЕВЕНТУАЛНА φ_0-УСТОЙЧИВОСТ НА ДИФЕРЕНЦИАЛНИ СИСТЕМИ ПО ОТНОШЕНИЕ НА ДВЕ МЕРКИ СЪС СМУТЕНИ ФУНКЦИИ НА ЛЯПУНОВ

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Резюме. Въвежда се понятие за евентуална φ_0-устойчивост на нелинейни системи от обикновени диференциални уравнения по отношение на две мерки. Техниката ни се опира на директния метод на Ляпунов. Приложени са смутени конусозначни функции на Ляпунов и са използвани обикновени скаларни диференциални уравнения.