ПЛОВДИВСКИ УНИВЕРСИТЕТ "ПАИСИЙ ХИЛЕНДАРСКИ", БЪЛГАРИЯ
НАУЧНИ ТРУДОВЕ, ТОМ 38, КН. 3, 2011 – МАТЕМАТИКА
PLOVDIV UNIVERSITY "PAISSII HILENDARSKI", BULGARIA
SCIENTIFIC WORKS, VOL. 38, BOOK 3, 2011 – MATHEMATICS

BASIC SUBGROUPS OF THE SYLOW P-SUBGROUPS OF SEMISIMPLE GROUP ALGEBRAS

Nako A. Nachev

Abstract. Let G be a separable abelian p-group with a basic subgroup B and let K be a field of a characteristic not equal to p where p is a prime number. In this paper we prove that the group $S(KB)$ is a basic subgroup of $S(KG)$.

Key words: commutative group algebras, unit groups, separable abelian p-groups, basic subgroups
Mathematics Subject Classification 2000: 16S34, 20C07

1. Introduction

The first papers about infinite group algebras are the articles of Berman [1-2], Berman-Rossa [3], Bovdi-Pataj [5], May[10], Mollov [12, 14-16], Nachev [10-21], Nachev-Mollov [22] and others. Let RG be the group algebra of an abelian group G over a commutative ring R with identity. Denote by tG the torsion subgroup of G, by G_p be the p-component of G, by $U(RG)$ the multiplicative group of RG and by $S(RG)$ the Sylow p-subgroup of the group $V(RG)$ of normalized units of RG, i.e. the p-component of $V(RG)$. The investigations of this group begin with the fundamental papers of Berman S. [1-2]) in which a complete description of $S(RG)$ (up to isomorphism) is given, when G is a countable abelian p-group and R is a countable field. Further Mollov T. [11-12] calculates the Ulm-Kaplansky invariants $f_\alpha(S)$ of the group $S(RG)$ when G is an arbitrary abelian group and R is a field of positive characteristic p. When R is a commutative ring with identity of prime characteristic p without nilpotent elements Bovdi A. and Pataj Z. [5] calculate the Ulm-Kaplansky invariants of $S(RG)$ under the following restriction: if the maximal divisible subgroup of G_p is not identity, then R is a p-divisible ring, i.e. $R^p = R$. Nachev N. and Mollov T. [22] calculate the invariants $f_\alpha(S)$ with the only restriction on G to be an abelian p-group. Nachev N. [21] calculates the invariants $f_\alpha(S)$ without restrictions on G and R. Moreover, in all indicated cases the authors give a full description, up to isomorphism, of the maximal divisible subgroup of $S(RG)$.

Let G be an abelian p-group and let K be a field whose characteristic is different from p. Berman S. and Rossa A. [3-4] have given a description of the torsion subgroup $tV(KG)$ of $V(KG)$ when G is a countable abelian p-group and K is a field. Let R be a commutative ring with identity, such that the characteristic of R does not divide the orders of the elements of G. N. Nachev [19-20] has given a description of the torsion subgroup $tV(RG)$ of $V(RG)$ when G is an abelian p-group and R contains the p^nth roots of unity, $n \in \mathbb{N}$. Karpilovsky G. [9, 5.2.5 Theorem, p.126] has determined the isomorphism class of $U(\mathbb{Q}G)$ when G is a finitely generated abelian group. Mollov T. [11-13, 15-17] has described the torsion subgroup $tV(RG)$ of $V(RG)$ when G is an abelian group and R is a field. Mollov T. [14] has also described $V(RG)$, up to isomorphism, when either

(a) G is an infinite direct sum of cyclic p-groups and $R = \mathbb{Q}$ or

(b) G is an abelian p-group and $R = \mathbb{R}$.

Chatzidakis Z. and Pappas P. [6] have determined the isomorphic class of $U(RG)$ when the torsion abelian group G is a direct sum of countable groups and R is a field. Nachev N. and Mollov N. [23-24] describe $U(RG)$, up to isomorphism, when G is an abelian p-group and at least one of the following conditions (a) or (b) is fulfilled:

(a) the first Ulm factor G/G^1 of G is a direct sum of cyclic groups and R is a field of the first kind with respect to p;

(b) R is a field of the second kind with respect to p.

If R is a direct product of m indecomposable rings $R_i,\ m \in \mathbb{N}$, Mollov T. and Nachev N. [18] give a description of the unit group $U(RG)$ of RG in the following cases:

(a) when R_i is a ring of prime characteristic p_i, tG/G_{p_i} is finite and the exponent of tG/G_{p_i} belongs to R_i^*;

(b) when R_i is of characteristic zero, R_i has no nilpotents, tG is finite of exponent n and $n \in R_i^*$.

Danchev P. [7] make an attempt to find the basic subgroup of $S(KG)$. However he has a difficulty to obtain this result.

This paper is organized as follows. Section 1 is an introduction. In Section

2 we prove some preliminary results. In Section 3 we formulate and give the main result.

The abelian group terminology is in agreement with [8].

2. Preliminary results

If G is an arbitrary additive abelian p-group, then the basic subgroup B of G is defined in the book of Fuchs [8]. Since the groups which we shall consider are multiplicative, then we have to change some concepts. For example, instead direct sum we shall use coproduct with a sign $\coprod$ and instead the equation $p^n x = a$ we shall use the notation $x^{p^n} = a$.

In the beginning we shall prove the following technical lemma.

Lemma 2.1. *Let G be an abelian group and let K be a field. Suppose H and F are finite subgroups of G, $H \bigcap F = 1$, $x \in KH$, $y \in KF$, $y \neq 0$ and $xy = 0$.*

Then $x = 0$.

Proof. Let $x = \sum_{h \in H} x_h h$ and $y = \sum_{f \in F} y_f f$. Then

$$xy = \sum_{h \in H} \sum_{f \in F} x_h \, y_f hf = 0.$$

Since $H \bigcap F = 1$, then the products hf form a group basis of the algebra $K(HF)$. Then the above formula implies $x_h \, y_f = 0$ for every $h \in H$ and $f \in F$. Since $y \neq 0$, then $y_f \neq 0$ for some $f \in F$ and $x_h \, y_f = 0$ implies $x_h = 0$ for every $h \in H$. Therefore, $x = 0$.

□

Let now G be an abelian p-group and $H \leq G$. We denote by $I(KG; H)$ the ideal of KG generated by the elements $h - 1, h \in H$. It is easily to see that the ideal $I(KG; H)$ coincides with the kernel of the homomorphism

$$(*) \qquad \varphi : KG \to K(G/H),$$

which is a continuation by an additivity of the natural homomorphism $G \to G/H$.

Lemma 2.2. *Let $\prod = \prod \big(G/H\big)$ be a transversal for H in G and*

$$(2.1) \qquad x = \sum_{g \in T} \sum_{h \in N} x_{gh} gh \in KG,$$

where T is a finite subset of $\prod$ and N is a finite subgroup of H. Then $x \in I(KG;H)$ if and only if for every $g \in T$

$$\sum_{h \in N} x_{gh} = 0 \tag{2.2}$$

holds.

Proof. Necessity. Let $x \in I(KG;H) = Ker\,\varphi$. Then $x\varphi = 0$ and (2.1) implies

$$x\varphi = \sum_{g \in T} \sum_{h \in N} x_{gh} gH = 0.$$

However the cosets $gH, g \in G$ form a K-basis of $K(G/H)$. Then the last equality implies (2.2).

Sufficiency. Let (2.2) holds for every $g \in T$. Then (2.1) imples $x\varphi = 0$. Hence $x \in I(KG;H)$.

□

Further we set

$$S(KG;H) = (1 + I(KG;H)) \bigcap S(KG). \tag{2.3}$$

It is easily to see that $S(KG;H)$ is a subgroup of $S(KG)$. The following lemma gives a description of $S(KG;H)$.

Lemma 2.3. *Let G be an abelian p-group, $H \leq G$ and let K be a field of characteristic different from p. Suppose $x \in S(KF)$, where F is a finite subgroup of G. Then $x \in S(KG;H)$ if and only if the following condition is fulfilled:*

$(**)$ *$xe_0 = e_0$, where e_0 is a minimal idempotent of $K(F \bigcap H)$ which corresponds to the identity character of $F \bigcap H$, i.e.*

$$e_0 = \left(1 \Big/ \left(F \bigcap H\right)\right) \sum_{f \in F \bigcap H} f.$$

Proof. Necessity. Let $x \in S(KG;H)$. We choose a transversal $\prod = \prod \left(F/\left(F \bigcap H\right)\right)$. Then the element x can be represented in the form

$$x = \sum_{g \in \Pi} \sum_{f \in F \bigcap H} x_{gf}\, gf, \quad x_{gf} \in R. \tag{2.4}$$

Since $fe_0 = e_0$ for every $f \in F \bigcap H$, then (2.4) implies

$$xe_0 = \sum_{g \in \Pi} \Big(\sum_{f \in F \bigcap H} x_{gf} \Big) g e_0. \tag{2.5}$$

However, $x \in S(KG; H)$. Hence

$$\sum_{f \in F \bigcap H} x_{gf} = \begin{cases} 1, & \text{if} \quad g = 1; \\ 0, & \text{if} \quad g \neq 1. \end{cases}$$

Then (2.5) implies $xe_0 = e_0$.

Sufficiency. Let $xe_0 = e_0$. Now we represent x in the form

$$x = xe_0 + x(1 - e_0).$$

Then $xe_0 = e_0$ implies

$$x = e_0 + x(1 - e_0). \tag{2.6}$$

Since $e_0 H = H$, then, with the applying of the homomorphism φ, given by the formula $(*)$, to the equality (2.6) we obtain

$$x\varphi = e_0 H + x(1 - e_0)H = H + x(H - H) = H.$$

Hence, $(x - 1)\varphi = 0$. Therefore, $x - 1 \in Ker\, \varphi = I(KG; H)$. In this way $x \in (1 + I(KG; H))$. However, $x \in S(KG)$.

Hence, $x \in (1 + I(KG; H)) \bigcap S(KG) = S(KG; H)$.

□

Lemma 2.4. *If $G = F \times H$, then $S(KG) = S(KF) \times S(KG; H)$.*

This lemma follows immediately from a result of Mollov and Nachev [18, Lemma 5.4].

3. Main result

A main goal of this section is to find a basic subgroup of $S(KG)$. We suppose that G is a separable abelian p-group with a basic subgroup B, $B = \coprod_{i=1}^{\infty} B_i$, where B_i is a coproduct of cyclic groups of orders p^i, $i \in \mathbb{N}$ and K is a field of the first kind with respect to p of characteristic not equal to p. Let κ be a constant of K with respect to p. We introduce first the following concept.

Definition 3.1 *A subgroup H of a group G is said to be a standard subgroup of the kind n, if*

1) $H = H_1 \times H_2$,

2) $H_1 \leq \coprod_{i=1}^{s} B_i$, $s = \max(\kappa + n - 1, 2n - 1)$,

3) $H_2 \leq M_s$, *where* M_s *is a maximal subgroup of* G *with a basic subgroup* $\coprod_{k=s+1}^{\infty} B_k$ *and*

4) $\exp H_1 \leq p^n$ *and* $\exp H_2 \leq p^n$.

Lemma 3.2 *If F is a finite subgroup of the group G and the exponent of F is p^n, then F can be put into a finite standard subgroup of the kind n of G.*

Proof. We project F on the subgroups $\coprod_{i=1}^{s} B_i$ and M_s. Let these projections be H_1 and H_2, respectively. Then $F \leq H = H_1 \times H_2$. Since F is finite, then the projections H_1 and H_2 are finite. Therefore, H is finite. The constructions of H, H_1 and H_2 implies that conditions 1), 2) and 3) of Definition 3.1 are fulfilled. Since the exponent of the projection of a group does not exceed the exponent of the same group, then condition 4) holds. However, at least one of the exponents of H_1 and H_2 will be exactly equal to p^n.

□

Now we can formulate the main result of the paper.

Theorem 3.3. *(Main result). Let G be a separable abelian p-group with a basic subgroup B and K be a field of the first kind with respect to p of characteristic not equal to p. Then $S(KB)$ is a basic subgroup of $S(KG)$.*

Proof. By the results of Mollov, $S(KB)$ is a pure subgroup of $S(KG)$ [17] and it is a coproduct of cyclic p-groups [15]. It is remain to prove that the quotient-group $S(KG)/S(KB)$ is divisible which is equivalent to

$$S(KG) = S(KB)S^p(KG). \tag{3.1}$$

Namely, we take an element x from the left hand of (3.1). We shall prove that x belongs to the right hand of (3.1).

We can suppose, by Lemma 3.2, that $x \in KH$, where H is a finite standard subgroup of the kind n of G. Let the components of H, by Definition 3.1, are H_1 and H_2. Then, by Lemma 2.4, x has the representation $x = x_1x_2$, where $x_1 \in S(KH_1) \subseteq S(KB)$ and $x_2 \in S(KH; H_2)$. It is remain to prove that $x_2 \in S^p(KG)$. To this aim we have to prove that if e is a minimal idempotent of the algebra KH which kernel does not contain H_2, then there exists an element y such that $xe = y^p$, where $y \in S(KGe)$. We explain that KG is considered as algebra with an identity e. Namely, we choose an element $h \in H_2$ such that $he \neq e$. There exists such element since $Ker\, e$ does not contain H_2. The order of h is not grater than p^n. Hence, the height of h in M_s is $\geq p^{s-n+1}$. In this way there exists an element $z \in M_s$ such that $h = z^{p^{s-n+1}}$. We set $H_3 = \langle H, z\rangle$. We decompose the idempotent e in a sum of minimal idempotents in KH_3. Let this decomposition is

$$e = e''_1 + e''_2 + \ldots + e''_t. \tag{3.2}$$

The group $S(KH_3e''_i)$ is a cyclic p-group for every $i = 1, 2, \ldots, t$ and contains two subgroups, namely $\langle xe''_i\rangle$ and $\langle ze''_i\rangle$. The order of the subgroup $\langle xe''_i\rangle$ is not greater than p^{s-n+1} and the order of $\langle ze''_i\rangle$ is greater than p^{s-n+1}. Consequently, $\langle xe''_i\rangle$ is strictly contained in $\langle ze''_i\rangle$. This inclusion implies the equality

$$xe''_i = z^{p\lambda_i}e''_i \tag{3.3}$$

for every $i = 1, 2, \ldots, t$, where λ_i is a integer. Now we set

$$y = z^{\lambda_1}e''_1 + z^{\lambda_2}e''_2 + \ldots + z^{\lambda_t}e''_t. \tag{3.4}$$

Then (3.2), (3.3) and (3.4) imply

$$xe = xe''_1 + xe''_2 + \ldots + xe''_t = z^{p\lambda_1}e''_1 + z^{p\lambda_2}e''_2 + \ldots + z^{p\lambda_t}e''_t = y^p,$$

i.e. $xe = y^p$. Besides, from (3.4), we have $y \in S(KH_3e) \leq S(KGe)$.

□

Remark. We note that Danchev P. [7] comments the question whether $S(KB)$ is a basic subgroup of $S(KG)$. He marks that the first two conditions for a basic subgroup are well known for $S(KB)$. He writes that the third condition, namely that $S(KG)/S(KB)$ is divisible group, is not known and he does not prove it in his paper. As it is seen from the proof of our Theorem 3.3 we prove exactly this condition.

Acknowledgments

Research was partially supported by the fund "NI" of Plovdiv University, Bulgaria under Contract FMI No. 43 (2009).

References

[1] Berman S., Group algebras of countable abelian p-groups, *Dokl. Akad. Nauk SSSR*, **175**, No. 3, (1967), 514–516, (In Russian).

[2] Berman S., Group algebras of countable abelian p-groups, *Publ. Math. Debrecen*, **14**, (1967), 365–405, (In Russian).

[3] Berman S., Rossa A., The Sylow p-subgroup of a group algebra of a countable abelian p-group, *Dopovidi Akad.Nauk. Ukrain. RSR*, Ser. A, 870-872, (1968), (In Ukrainian).

[4] Berman S., Rossa A., The Sylow p-subgroups of the group algebras of countable abelian p-groups, *Proc. of the XXIX Scientific Conference of Professors and Instructors of the Staff of the Uzhgorod University, Dept. of Mathematical Sciences*, Uzhgorod, (1975), 158–176, (In Russian).

[5] Bovdi A., Pataj Z., On constructing a multiplicative group of a group ring of a p-group over a ring of characteristic p (Russian), *Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauka*, No. 1, (1978), 5–11, (In Russian).

[6] Chatzidakis Z., Pappas P., Units in abelian group rings, *J. London Math. Soc.* (2) **44**, (1991), 9–23.

[7] Danchev P., Sylow p-subgroups of abelian group rings, *Serdica Math. J.*, **29**, (2003), 33–44.

[8] Fuchs L., *Infinite abelian groups*, Vol. I and II, New York: Academic Press, (1970 and 1973).

[9] Karpilovsky G. , *Commutative group algebras*, Marcel Dekker, Inc., New York, (1983).

[10] May W., Modular group algebras of simply presented abelian groups, *Proc. Amer. Math. Soc.*, **104** (No. 2), (1988), 403–409.

[11] Mollov T., *Ulm invariants of the Sylow p-subgroups of the group algebras of the abelian groups over a field of characteristic p*, Sixth Congress of the Bulgarian Mathematicians, Varna, Reports Abstracts, Section A2, 2-2, (1977), (In Russian).

[12] Mollov T., Ulm invariants of the Sylow p-subgroups for the group algebras of the abelian groups over a field of characteristic p, *PLISKA, Stud. Math. Bulg.*, **2**, (1981), 77–82, (In Russian).

[13] Mollov T., On multiplicative groups of semisimple group algebras of abelian p-groups, *C. R. Acad. Bulgar. Sci.* **35**, (1982), 1619–1622, (In Russian).

[14] Mollov T., On multiplicative groups of real and rational group algebras of abelian p-groups, *C. R. Acad. Bulgar. Sci.*, **37**, (1984), 1151–1153, (In Russian).

[15] Mollov T., Sylow p-subgroups of the group of the normalized units of semisimple group algebras of uncountable abelian p-groups, *Pliska Stud. Math. Bulgar.* **8**, (1986), 34–46, (In Russian).

[16] Mollov T., Multiplicative groups of semisimple group algebras, *Pliska Stud. Math. Bulgar.* **8**, (1986), 54–64, (In Russian).

[17] Mollov T., Ulm–Kaplansky invariants of Sylow p-subgroups of normalized units of semisimple group algebras of infinite separable abelian p-groups, *Pliska Stud. Math. Bulgar.* **8**, (1986), 101–106, (In Russian).

[18] Mollov T., Nachev N., Unit groups of commutative group rings, *Communications in Algebra*, **34**, (2006), 3835–3857.

[19] Nachev N., Ulm–Kaplansky invariants of the group of the normalized units of group algebras of abelian p-groups over a commutative ring in which p is an unit, *C. R. Acad. Bulgar. Sci.* **33**, (1980), 1585–1587, (In Russian).

[20] Nachev N., Ulm–Kaplansky invariants of the group of the normalized units of group algebras of abelian p-groups over a commutative ring in which p is an unit, *Pliska Stud. Math. Bulgar.* **8**, (1986), 21–33, (In Russian).

[21] Nachev N., Invariants of the Sylow p-subgroup of the unit group of a commutative group ring of characteristic p, *Communications in Algebra*, **23** (7), (1995), 2469–2489.

[22] Nachev N., Mollov T., Ulm-Kaplansky invariants of the normed unit groups of the modular group rings of a primary abelian group, *Serdica Math. J.*, **6**, (1980), 258–263, (In Russian).

[23] Nachev N., Mollov T., Unit groups of semisimple group algebras of abelian p-groups over field, *C. R. Acad.Bulgar. Sci.*, **46**, (1993), 17–19.

[24] Nachev N., Mollov T., Unit groups of semisimple group algebras of abelian p-groups over a field, *J. Algebra*, **188**, (1997), 580–589.

Department of Algebra
University of Plovdiv
4000 Plovdiv, Bulgaria
e-mail: nachev@uni-plovdiv.bg

Received 23 November 2011

БАЗИСНИ ПОДГРУПИ НА СИЛОВСКИТЕ P-ПОДГРУПИ НА ПОЛУПРОСТИ ГРУПОВИ АЛГЕБРИ

Нако А. Начев

Резюме. Нека G е сепарабелна абелева p-група с базисна подгрупа B и нека K е поле с характеристика, различна от p, където p е просто число. В тази статия се доказва, че групата $S(KB)$ е базисна подгрупа на $S(KG)$.