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L_P-EQUIVALENCE BETWEEN TWO ORDINARY IMPULSE DIFFERENTIAL EQUATIONS WITH BOUNDED LINEAR IMPULSE OPERATORS IN A BANACH SPACE

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Abstract. By the help of the fixpoint principle of Schauder-Tychonoff and Banach are found sufficient conditions for the existence of L_p-equivalence between two ordinary impulse differential equations with bounded linear impulse operators in an arbitrary Banch space.

Key words: Impulse Differential Equations, L_p-equivalence, Schauder-Tychonoff and Banach fixpoint principle
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1. Introduction

In the paper we study the L_p-equivalence between two ordinary impulse differential equations with bounded linear impulse operators in an arbitrary Banch space. This means, that to every solution of the first equation, which lies in a closad and convex set, there corresponds a soluition of the second equation, which lies in an other closed and convex set and the difference between both solutions lies in the spaces L_p and vice versa. In Theorem 1. and Theorem 2. are found sufficient conditions for the existence of L_p-equivalence between the considered equations.

2. Problem statement

Let X is an arbitrary Banach space with norm $||.||$ and identity I. Let $\mathbb{R}_+ = [0, +\infty)$. By $\{t_n\}_{n=1}^{\infty}$ we shall denote a sequence of points

$$0 = t_0 < t_1 < t_2 < ... < t_n < ... \text{ ,satisfying the condition } \lim_{n\to\infty} t_n = \infty.$$

We consider the following impulse differential equation:

$$\frac{du_i}{dt} = F_i(t, u_i) \quad \text{for } t \neq t_n \tag{1}$$

$$u_i(t_n^+) = Q_n^i(u_i(t_n)) \quad \text{for } n = 1, 2, ... \tag{2}$$

where $F_i(.,.) : \mathbb{R}_+ \times X \to X \ \ (i = 1, 2)$ are continuous functions and $Q_n^i : X \to X (i = 1, 2; n = 1, 2, ...)$ are linear bounded operators. Furthermore, we assume that all considered functions are continuous from the left. Let $Q_0^i = I$ and let

$$w_i(t, s) = \prod_{s \le t_j \le t} Q_j^i \quad (i = 1, 2;\ 0 \le s \le t)$$

Lemma 1. *The solutions $u_i(t)(i = 1, 2)$ of the integral equations*

$$u_i(t) = w_i(t, 0)u_i(0) + \int_0^t w_i(t, s)F_i(s, u_i(s))ds \qquad (i = 1, 2) \tag{3}$$

satisfy the impulse differential equations (1), (2) (i=1,2).

ProofLet $t \in (t_n, t_{n+1})$. Then

$$\begin{aligned} u_i(t) = w_i(t, 0)u_i(0) + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} w_i(t, s)F_i(s, u_i(s))ds+ \\ + \int_{t_n}^{t} w_i(t, s)F_i(s, u_i(s))ds = \\ = \prod_{j=1}^{n} Q_j^i u_i(0) + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \prod_{j=k+1}^{n} Q_j^i F_i(s, u_i(s))ds + \int_{t_n}^{t} F_i(s, u_i(s))ds \end{aligned}$$

We differentiate in respect to t and receive

$$\frac{du_i}{dt} = \frac{d}{dt}(\prod_{j=1}^{n} Q_j^i u_i(0)) + \sum_{k=0}^{n-1} \frac{d}{dt}(\int_{t_k}^{t_{k+1}} \prod_{j=k+1}^{n} Q_j^i F_i(s, u_i(s))ds) +$$
$$+ \frac{d}{dt}(\int_{t_n}^{t} F_i(s, u_i(s))ds) = F_i(t, u_i(t))$$

Let $t = t_n$. Then

$$u_i(t_n^+) = w_i(t_n^+, 0)u_i(0) + \int_0^{t_n^+} w_i(t_n^+, s)F_i(s, u_i(s))ds =$$
$$= \prod_{j=1}^{n} Q_j^i u_i(0) + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \prod_{j=k+1}^{n} Q_j^i F_i(s, u_i(s))ds =$$
$$= Q_n^i \prod_{j=1}^{n-1} Q_j^i u_i(0) + Q_n^i \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \prod_{j=k+1}^{n-1} Q_j^i F_i(s, u_i(s))ds =$$
$$= Q_n^i(u_i(t_n))$$

Hence for $t = t_n$ $(n = 1, 2, ...)$ the solutions $u_i(t)$ $(i = 1, 2)$ of the integral equations (3) $(i = 1, 2)$ satisfy the jump condition (2) $(i = 1, 2)$. □

By $L_p(X)$, $1 \le p < \infty$ we denote the space of all functions $u : \mathbb{R}_+ \to X$ for which $\int_0^\infty \|u(t)\|^p dt < \infty$ with norm $\|u\|_p = (\int_0^\infty \|u(t)\|^p dt)^{\frac{1}{p}}$.

Definition 1. *([3]) The equation* (1), (2) *for* $i = 2$ *is called* L_p*-equivalent to the equation* (1), (2) *for* $i = 1$ *in the unempty, closed and convex subset* B *of* X*, if there exists convex and closed subset* D *of* X*, such that for any solution* $u_1(t)$ *of* (1), (2) $(i = 1)$ *lying in the set* B *there exists a solution* $u_2(t)$ *of* (1), (2) $(i = 2)$ *lying in the set* $B \cup D$ *and satisfying the relation* $u_2(t) - u_1(t) \in L_p(X)$*. If the equation* (1), (2) $(i = 2)$ *is* L_p*-equivalent to the equation* (1), (2) $(i = 1)$ *in the set* B *and vice versa, we shall say that equations* (1), (2) $(i = 1)$ *and* (1), (2) $(i = 2)$ *are* L_p*-equivalent in the set* B*.*

Let $S(\mathbb{R}_+, X)$ is the linear set of all functions which are continuous for $t \neq t_n$ $(n=1, 2, ...)$, have both left and right limits at points t_n and are continuous from the left. The set $S(\mathbb{R}_+, X)$ is a locally convex space w.r.t. the metric

$$\rho(u, v) = \sup_{0<T<\infty} (1 + T)^{-1} \frac{\max_{0\le t\le T} \|u(t) - v(t)\|}{1 + \max_{0\le t\le T} \|u(t) - v(t)\|}.$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli's theorem is valid.

Lemma 2. *([1]) The set $H \subset S(\mathbb{R}_+, X)$ is relatively compact if the intersections $H(t) = \{h(t) : \ h \in H\}$ are relatively compact for $t \in \mathbb{R}_+$ and H is equicontinuous on each interval $(t_n, t_{n+1}]$ $(n = 0, 1, 2, ...)$.*

ProofWe apply Arzella-Ascoli's theorem on each intervals
$(t_n, t_{n+1}]$ $(n = 0, 1, 2, ...)$ and constitute a diagonal line sequence, which is converging on each of them. □

Let C is an unempty subset of X and let

$$\tilde{C} = \{u \in S(\mathbb{R}_+, X) : u(t) \in C, t \in \mathbb{R}_+\}$$

Lemma 3. *([3]) Let C is an unempty, convex and closed subset of X and the operator T is continuous, compact and maps $\tilde{C}$ into itself.*
Then T has a fixpoint in $\tilde{C}$.

ProofIt follows from the fixpoint principles of Schauder-Tychonoff. □

3. Main results

Let $u(t) = u_2(t) - u_1(t)$, where $u_i(t)$ $(i = 1, 2)$ are defined by (3). Then the function $u(t)$ is a solution of the integral equation $u(t) = T(u_1, u)(t)$, where

$$\begin{aligned} T(u_1, u)(t) &= w_2(t, 0)(u_1(0) + u(0)) - w_1(t, 0)u(0) + \\ &+ \int_0^t (w_2(t, s)F_2(s, u_1(s) + u(s)) - w_1(t, s)F_1(s, u_1(s)))ds \end{aligned} \tag{4}$$

Now we will find sufficient conditions for the existence of L_p-equivence between the impulse differential equations (1), (2) $(i = 1, 2)$.

We shall prove, that for any solution $u_1(t)$ of the equation (1), (2) for $i = 1$, which lies in an unempty, closed and convex subset B of X, there exists a closed and convex subset D of X, where the operator $T(u_1, u)$ has such a fixpoint $u(t)$, that $u_1(t) + u(t) \in B \cup D$ and $u \in L_p(X)$.

Theorem 1. *Let the following conditions are fulfilled:*
1. There exists an unempty, convex and closed subset D of X, such that $T(u_1, u)(t) \in D$ for each u with $u(t) \in D$ $(t \in \mathbb{R}_+)$
2. The operator-functions $w_i(t, s)$ $(i = 1, 2)$ satisfy the conditions:

2.1. $||w_i(t,s)\xi|| \leq M||\xi|| \quad (i=1,2; \xi \in X; 0 \leq s < t < \infty)$, *where* M *is a positive number.*

2.2. $||w_2(t,0)\xi - w_1(t,0)\eta|| \leq \chi(t) \quad (0 \leq t < \infty)$, *where* $\xi \in \tilde{B} \cup \tilde{D}$, $\eta \in \tilde{B}$, $\chi \in L_p(\mathbb{R}_+)$.

3. The functions $F_i(t,v)$ *and* $w_i(t,s)$ $(i=1,2)$ *satisfy the conditions:*

3.1. Fulfilled is

$$\sup_{v \in \tilde{B}, w \in \tilde{B} \cup \tilde{D}} \int_0^t ||w_2(t,s)F_2(s,w) - w_1(t,s)F_1(s,v)||ds \leq \psi(t),$$

where $\psi \in L_p(\mathbb{R}_+)$.

3.2. For any fixed $u_1 \in \tilde{B}$ *the following inclusions hold*

$$\int_0^t w_2(t,s)F_2(s, u_1(s) + u_2(s))ds \in K^{u_1}(t),$$

where $K^{u_1}(t)$ *is for any fixed* $t \in \mathbb{R}_+$ *a compact subset of* X.

3.3. Fulfilled is

$$\sup_{w \in \tilde{B} \cup \tilde{D}} ||F_2(t,w)|| \leq \phi(t),$$

where $\phi(t)$ *is integrable function for each interval* $(t_n, t_{n+1}] \quad (n = 0,1,2,...)$.

Then the equation $(1),(2)$ *for* $i=2$ *is* L_p*-equivalent to the equation* $(1),(2)$ *for* $i=1$ *in the set* B.

ProofFrom condition 1. of Theorem 1 it follows, that the operator $T(u_1,u)$ defined by (4) maps the set $\tilde{D} = \{u \in S(\mathbb{R}_+, X) : u(t) \in D, t \in \mathbb{R}_+\}$ into itself for $u_1 \in \tilde{B}$.

For each $u_1 \in \tilde{B}$ we set $H_{u_1} = \{h(t) = T(u_1,u)(t) : u \in \tilde{D}, t \in \mathbb{R}_+\}$. We will show the equicontinuity of the functions of the set H_{u_1}. Let $t' > t''$ and $t', t'' \in (t_n, t_{n+1}]$. Then

$$w_i(t',s) = w_i(t'',s) = \prod_{s \leq t_j < t'} Q_j^i \quad (i=1,2)$$

and hence

$$\begin{aligned}
&||h(t') - h(t'')|| \leq \\
&\quad \leq ||w_2(t',0)(u_1(0)+u(0)) - w_1(t',0)u_1(0) - \\
&\quad - w_2(t'',0)(u_1(0)+u(0)) + w_1(t'',0)u_1(0)|| + \\
&\quad + || \int_0^{t'} (w_2(t',s)F_2(s,u_1(s)+u(s)) - w_1(t',s)F_1(s,u_1(s)))ds - \\
&\quad - \int_0^{t''} (w_2(t'',s)F_2(s,u_1(s)+u(s)) - w_1(t'',s)F_1(s,u_1(s)))ds|| \leq \\
&\quad \leq \int_{t''}^{t'} ||w_2(t',s)F_2(s,u_1(s)+u(s)) - w_1(t',s)F_1(s,u_1(s))||ds \leq \\
&\quad \leq \int_{t''}^{t'} ||w_2(t',s)F_2(s,u_1(s)+u(s))||ds + \int_{t''}^{t'} ||w_1(t',s)F_1(s,u_1(s))||ds \leq \\
&\quad \leq M \sup_{w\in\tilde{B}\cup\tilde{D}} \int_{t''}^{t'} ||F_2(s,w)||ds + M \sup_{v\in\tilde{B}} \int_{t''}^{t'} ||F_1(s,v)||ds
\end{aligned}$$

From this estimate and from the continuosity of the functions $F_i(t,v)$ $(i=1,2)$ follows the equicontinuity of the functions of the set H_{u_1}.

From condition 3.2. and (4) follows, that the sets $H_{u_1}(t) = \{h(t) : h \in H_{u_1}\}$ are relatively compact for every $t \in \mathbb{R}_+$. From Lemma 2. follows the relatively compactness of the set H_{u_1}.

Now we will show, that the operator $T(u_1,u)$ is continuous in $S(\mathbb{R}_+,X)$.

Let the sequence $\{\tilde{u}_k\} \subset \tilde{D}$ is convergent in the metric of the space $S(\mathbb{R}_+,X)$ to the function $\tilde{u} \in \tilde{D}$.

Then from the continuity of the function $F_2(t,v)$ follows, that for $t \in \mathbb{R}_+$ the sequence $F_2(t,u_1(t)+\tilde{u}_k(t))$ convergence to $F_2(t,u_1(t)+\tilde{u}(t))$.

From conditions 2.1. and 3.3. follows, that the sequence of functions $w_2(t,s)F_2(s,u_1(s)+\tilde{u}_k(s))$ is bounded by an integrable function. Indded

$$||w_2(t,s)F_2(s,u_1(s)+\tilde{u}_k(s))|| \leq M \sup_{w\in\tilde{B}\cup\tilde{D}} ||F_2(s,w)|| \leq M\phi(s)$$

From the Lebesgue's Theorem follows, that in the untegral formula

$$\begin{aligned}
T(u_1,\tilde{u}_k)(t) &= w_2(t,0)(u_1(0)+\tilde{u}_k(0)) - w_1(t,0)u_1(0) + \\
&\quad + \int_0^t w_2(t,s)F_2(s,u_1(s)+\tilde{u}_k(s))ds - \int_0^t w_1(t,s)F_1(s,u_1(s))ds
\end{aligned}$$

is possible to go to the limit. Hence $T(u_1, \tilde{u}_k)(t)$ converges to $T(u_1, \tilde{u})(t)$ for $t \in \mathbb{R}_+$. From this convergence and from the compactness of the operator $T(u_1, u)$ follows the convergence in $S(\mathbb{R}_+, X)$.

From Lemma 3. follows, that for every $u_1 \in \tilde{B}$ the operator $T(u_1, u)$ has a fixpoint $u \in \tilde{D}$ i.e. $u = T(u_1, u)$.

Now we will show, that this fixpoint lies in $L_p(X)$. From conditions 2.2., 3.1. and (4) we receive

$$||u(t)|| = ||w_2(t,0)(u_1(0) + u(0)) - w_1(t,0)u_1(0)||+ \\ + \sup_{v \in \tilde{B}, w \in \tilde{B} \cup \tilde{D}} \int_0^t ||w_2(t,s)F_2(s,w) - w_1(t,s)F_1(s,w)||ds \leq \chi(t) + \psi(t)$$

Then from the inequality of Minkowski follows

$$||u||_p \leq ||\chi + \psi||_p \leq ||\chi||_p + ||\psi||_p$$

Hence the equation (1), (2) $(i = 2)$ is L_p-equivalent to the equation (1), (2) $(i = 1)$ in the set B. □

Remark 1. *The case, when $dim X < \infty$ and the sets B and D are closed balls with center zero is considered in [2]. In this case the condition 3.2. of Theorem 1. is automaticaly fulfilled.*

Now by the help of the Banach fixpoint principle we will prove, that for every fixed $u_1 \in \tilde{B}$ the operator $T(u_1, u)$ has an unique fixpoint $u \in \tilde{D}$, such that $u \in L_p(X)$.

Theorem 2. *Let the following conditions are fulfilled:*

1. The conditions 1., 2.2. and 3.1. of Theorem 1.

2. The operator-function $w_2(t,s)$ $(0 \leq s < t < \infty)$ fulfilled the condition

$$||w_2(t,s)\xi|| \leq M||\xi||,$$

where M is a positive number and $\xi \in X$.

3. The function $F_2(t,v)$ satisfies the condition

$$||F_2(t,v) - F_2(t,w)|| \leq \psi(t)||v - w||,$$

where $\psi(.) : \mathbb{R}_+ \to \mathbb{R}_+$ and $v, w \in \tilde{B} \cup \tilde{D}$.

4. The function $\psi(t)$ and the constant M from condition 2. satisfy the condition

$$M(1 + \int_0^\infty \psi(s)ds) < 1.$$

Then for every fixed $u_1 \in \tilde{B}$ *the operator* $T(u_1, u)$ *has an unique fixpoint* $u \in \tilde{D}$.

ProofLet $u_1 \in \tilde{B}$ is fixed and $u', u'' \in \tilde{D}$. We denote

$$|||u' - u''||| = \max_{t \in \mathbb{R}_+} ||u'(t) - u''(t)||.$$

From conditions 2. and 3. we obtain

$$\begin{aligned}
&||T(u_1, u')(t) - T(u_1, u'')(t)|| \leq \\
&\leq ||w_2(t, 0)(u_1(0) + u'(0)) - w_2(t, 0)(u_1(0) + u''(0))||+ \\
&+ \int_0^t ||w_2(t, s)F_2(s, u_1(s) + u'(s)) - w_2(t, s)F_2(s, u_1(s) + u''(s))||ds \leq \\
&\leq M||u'(0) - u''(0)|| + M \int_0^t ||F_2(s, u_1(s) + u'(s)) - F_2(s, u_1(s) + u''(s))||ds \leq \\
&\leq M|||u' - u''||| + M \int_0^t \psi(s)||u'(s) - u''(s)||ds \leq \\
&\leq (M + M \int_0^t \psi(s)ds)\ |||u' - u''|||
\end{aligned}$$

From condition 4. and the last estimate follows, that the operator $T(u_1, u)$ is a contraction. □

Remark 2. *Similar problems for the existence of* L_p*-equivence between nonlinear impulse differential equations with unbounded linear parts in an arbitrary Banch space are considered in [3] and [4]. The case, when the linear parts are bounded operators in Banach space is considered in [5], [6]. Sufficient conditions for the existence of* L_p*-equivence between impulse differential equations in N-dimensional Euclidean space are found in [2].*

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L_P-ЕКВИВАЛЕНТНОСТ МЕЖДУ ДВЕ ОБИКНОВЕНИ ИМПУЛСНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ С ОГРАНИЧЕНИ ЛИНЕЙНИ ИМПУЛСНИ ОПЕРАТОРИ В БАНАХОВО ПРОСТРАНСТВО

Георги Костадинов, Андрей Захариев

Резюме. С помощта на принципите за неподвижната точка на Шаудер-Тихонов и Банах са намерени достатъчни условия за съществуването на L_p-еквивалентност между две обикновени импулсни диференциални уравнения с ограничени линейни импулсни оператори в произволно Банахово пространство.